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On nonlinear scattering of drift wave by toroidal Alfvén eigenmode in tokamak plasmas

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Abstract

Using electron drift wave (eDW) as a paradigm model, we have investigated analytically direct wave–wave interactions between a test DW and ambient toroidal Alfvén eigenmodes (TAEs) in toroidal plasmas, and their effects on the stability of the eDW. The nonlinear effects enter via scatterings to short-wavelength electron Landau damped kinetic Alfvén waves (KAWs). Specifically, it is found that scatterings to upper-sideband KAW lead to stimulated absorption of eDW. Scatterings to the lower-sideband KAW, on the contrary, lead to its spontaneous emission. As a consequence, for typical parameters and fluctuation intensity, nonlinear scatterings by TAEs have negligible net effects on the eDW stability; in contrast to the 'reverse' process investigated in Chen *et al* (2022 *Nucl. Fusion* **62** 094001), where it is shown that nonlinear scattering by ambient eDWs may lead to significant damping of TAE.

Keywords: toroidal Alfvén eigenmode, burning plasma, drift wave, nonlinear mode coupling, gyrokinetic theory

(Some figures may appear in colour only in the online journal)

1. Introduction

Drift wave (DW) [1] and shear Alfvén wave (SAW) [2–4] are two fundamental electromagnetic oscillations in magnetized plasmas such as tokamaks. DWs are, typically, electrostatic fluctuations excited by thermal plasma density and/or temperature nonuniformities. Consequently, DWs have frequencies, perpendicular wavelengths and parallel wavelengths comparable, respectively, to the thermal plasma diamagnetic drift frequencies, thermal ion Larmor radii and the system size. SAWs, meanwhile, are electromagnetic fluctuations and, typically, manifest themselves as Alfvén eigenmodes (AEs) located within the frequency gaps of SAW continuous spectra [2]. For typical tokamak parameters, AE frequencies could be an order of magnitude higher than those of DWs, and, thus, spontaneous excitations of AEs often involve resonances with superthermal energetic particles (EPs); e.g. alphas in a D-T fusion plasma. AEs, thus, have perpendicular wavelengths in the order of EP Larmor radii and parallel wavelengths in the order of system size. In short, we may describe DWs as low-frequency micro-scale fluctuations; while AEs are meso-scale fluctuations at higher frequencies but still much lower than the ion cyclotron frequency. Since both DWs and AEs are intrinsic fluctuations in magnetic confined fusion plasmas and have routinely been observed in tokamak plasmas, it is, thus, natural

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to inquire whether and how these two kinds of fluctuations may interact and what the potential implications of these cross-scale interactions could be. Recently, we have investigated such interactions via the channel of nonlinear wave scatterings between toroidal Alfvén eigenmode (TAE) [2] and, as a paradigm model, electron drift wave (eDW). Interactions of DW turbulence and AEs have attracted significant interest in the recent years due to observed stabilization of tokamak turbulence by fast ions [5, 6]. However, some fundamental aspects remain to be clarified and understood concerning the underlying physics processes.

One important aspect when extrapolating from present day devices to reactor relevant fusion plasmas is the EP characteristic energy and normalized orbit width, which are responsible of remarkably different EP dynamic responses in the two cases [4]. Another aspect concerns whether the predominant cross-scale coupling process is direct or indirect. In the first group is either stimulated or spontaneous wave–wave coupling. Of the second type are processes mediated by zonal structures [7], e.g. zonal flows and fields [8, 9], including phase space zonal structures [10, 11].

An example of direct coupling is the TAE/ion temperature gradient (ITG) induced scattering, where EP may excite TAE by inverse ion Landau damping in the presence of finite amplitude ITG turbulence [12]. This mechanism has been invoked to explain the observed excitation of marginally stable TAE in gyrokinetic simulations of ITG [13], which then enhance the level of zonal flows and eventually yield to an appreciable reduction of ITG induced turbulence transport [14]. In this work, we further explore the DW-AE direct coupling channel via nonlinear wave scatterings using the eDW paradigm [15] with the aim of developing a comprehensive gyrokinetic description of theses precesses and of gaining insights into their possible impact on turbulent transport.

There are two types of direct nonlinear interactions between TAE and eDW. The first type involves the scattering of a test TAE by ambient eDWs [15]. In this case, it was demonstrated that the TAE will suffer significant damping via nonlinearly generated upper and lower sidebands of short-wavelength electron Landau damped kinetic Alfvén waves (KAWs) [16]. This scattering process, thus, may be regarded as stimulated absorption. Furthermore, for typical parameters, it is found that the nonlinear damping rate could be comparable to the growth rate of TAE instability excited by EPs. The second type of nonlinear wave–wave interactions involve the scattering of a test eDW by ambient TAEs, and is the actual focus of the present work. As will be shown in the following analysis, while the second type of scattering may be considered as the 'reverse' of the first one, the induced nonlinear damping/growth rate in this case is, in fact, negligible for typical parameters. Qualitatively speaking, while the nonlinearly generated upper sideband KAW (UKAW) still gives rise to stimulated absorption, the nonlinearly generated lower sideband KAW (LKAW), however, gives rise to spontaneous emission (i.e. as in a parametric decay instability) [17]. Quantatively, these two effects tend to nearly cancel each other; leading to negligible net effect on the stability of eDW.

The plan of this work is as follows. The theoretical model and governing equations are given in section 2. Section 3 discusses the nonlinear generation of upper and lower KAW sidebands. Nonlinear dispersion relation of eDW in the presence of the finite-amplitude TAE is then derived and analyzed in section 4. Section 5 gives the final conclusions and discussions.

2. Theoretical model and governing equations

We consider a large-aspect-ratio and low-β tokamak plasma with circular magnetic surfaces. Thus, $\epsilon \equiv r/R \ll 1$ with r and R being, respectively, the minor and major radii of the torus, and $\beta \sim O(\epsilon^2) \ll 1$ being the ratio between plasma and magnetic pressure. We, furthermore, take the thermal background plasma to be Maxwellian, and adopt the eDW paradigm model with finite density gradient but negligible temperature gradient as well as trapped particle effects in order to simplify the theoretical analyses and, thereby, illuminate the underlying physics.

The perturbed distribution function, δf_j with $j = e, i$

$$\delta f_j = -(e/T)_j F_{Mj} \delta\phi + \exp(-\boldsymbol{\rho} \cdot \nabla)\delta g_j, \tag{1}$$

obeys the nonlinear gyrokinetic equation [18]

$$\left(\partial_t + v_\| \mathbf{b} \cdot \nabla + \mathbf{v}_d \cdot \nabla + \langle\delta\mathbf{u}_g\rangle_\alpha\cdot\right)\delta g_j \\ = (e/T)_j F_{Mj} \left(\partial_t + i\omega_{*j}\right)\langle\exp(\boldsymbol{\rho}_j \cdot \nabla)\delta L\rangle_\alpha. \tag{2}$$

Here, F_{Mj} is the Maxwellian distribution, $\boldsymbol{\rho}_j = \mathbf{b} \times \mathbf{v}/\Omega_j$, $\mathbf{b} \equiv \mathbf{B}_0/B_0$, $\Omega_j = (eB_0/mc)_j$, δg_j is the non-adiabatic particle response, $\mathbf{v}_d = \mathbf{b} \times [(v_\perp^2/2)\nabla \ln B_0 + v_\|^2 \mathbf{b} \cdot \nabla\mathbf{b}]$ is the magnetic drift velocity, $\langle\mathbf{A}\rangle_\alpha$ denotes the gyro-phase averaging of $\mathbf{A}$, $\omega_{*j} = -i(cT/eB_0)\mathbf{b} \times \nabla \ln N_j \cdot \nabla$ is the diamagnetic drift frequency due to the finite density gradient,

$$\langle\delta\mathbf{u}_j\rangle_\alpha = (c/B_0)\mathbf{b} \times \nabla\langle\exp(-\boldsymbol{\rho}_j \cdot \nabla)\delta L\rangle_\alpha, \tag{3}$$

and

$$\delta L = \delta\phi - v_\| \delta A_\|/c \tag{4}$$

with $\delta\phi$ and $\delta A_\|$ being, respectively, the scalar and parallel component of the vector potential. Note that, with $\beta \ll 1$, magnetic compression may be neglected; i.e. $\delta B_\| \simeq 0$.

Meanwhile, the governing field equations are the quasi-neutrality condition

$$\sum_{j=e,i} \left[(N_0 e^2/T)_j \delta\phi - e_j \langle(J_k \delta g)_j\rangle_v\right] = 0, \tag{5}$$

and the parallel Ampere's law $\nabla_\perp^2 \delta A_\| = -(4\pi/c)\delta J_\|$. Here, we note $J_k = J_0(k_\perp\rho) = \langle\exp(i\boldsymbol{\rho} \cdot \mathbf{k}_\perp)\rangle_\alpha$ and $k_\perp^2 = -\nabla_\perp^2$ should be understood as an operator. Furthermore, we note that, for SAW and KAW, instead of the Ampere's law, it is more convenient to use the following nonlinear gyrokinetic vorticity equation [19, 20]
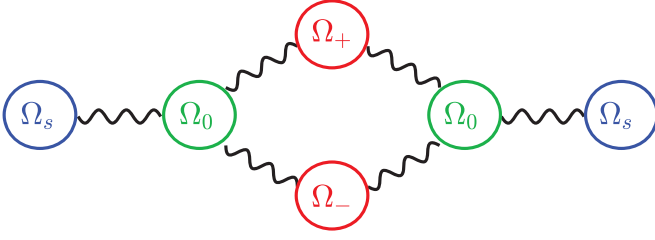
Figure 1. Schematic diagram of the two-step scattering processes analyzed in the present work. The test eDW, ambient TAE and nonlinearly generated KAW sidebands are in blue, green and red, respectively.

$$i k_\| \delta J_{\|k} + (N_0 e^2/T)_i \, (1 - \Gamma_k)(\partial_t + i\omega_{*i})_k \, \delta\phi_k$$
$$+ \, i \sum_j \langle e_j J_k \omega_d \delta g_j \rangle_v = \sum_{\mathbf{k}' + \mathbf{k}'' = \mathbf{k}} \Lambda_{k''}^{k'} \left\{ \delta A_{\|k'} \delta J_{k''}/c \right.$$
$$\left. - e_j \langle (J_k J_{k'} - J_{k''}) \delta L_{k'} \delta g_{k''j} \rangle_v \right\}. \quad (6)$$

Here, $\Gamma_k \equiv I_0(b_k) \exp(-b_k)$, $b_k = k_\perp^2 \rho_i^2$, $\rho_i^2 = T_i/(m_i \Omega_i^2)$, $\omega_d = \mathbf{k}_\perp \cdot \mathbf{v}_d$ and I_0 is the modified Bessel function. The first and second terms on the left hand side correspond, respectively, to the field line bending and inertia terms. Meanwhile, the third term corresponds to the curvature-pressure coupling term including the ballooning-interchange term and finite plasma compression. Note that, for TAE/KAW physics considered here, it can generally be ignored. The right hand side contains the nonlinear terms, where $\Lambda_{k''}^{k'} = (c/B_0)\mathbf{b} \cdot (\mathbf{k}'' \times \mathbf{k}')$, and the first and second terms correspond, respectively, to the Maxwell and generalized gyrokinetic ion Reynolds stresses. Note that, since eDW is predominantly electrostatic, the Maxwell stress makes negligible contribution in the present analysis.

We now consider the effects on eDW linear stability due to nonlinear scattering by TAE. Letting $\Omega_0(\omega_0, \mathbf{k}_0)$ and $\Omega_s = (\omega_s, \mathbf{k}_s)$ denote, respectively, a small but finite-amplitude TAE with toroidal mode number, n_0, and a test eDW with toroidal mode number, n_s. Thus, $|\omega_0| \simeq V_A/(2qR)$ with V_A being the Alfvén speed and q the safety factor, $\omega_s \sim \omega_{*e}$ the electron diamagnetic drift frequency, and $|k_{s\theta}\rho_i| = |n_s q \rho_i/r| \sim O(1)$. Furthermore, we have, typically, $|\omega_s/\omega_0| \ll 1$ and $|n_0/n_s| \ll 1$. That is, TAE and eDW are disparate both in spatial and temporal scales. Consequently, the sidebands nonlinearly generated by TAE and eDW; i.e. $\Omega_\pm = (\omega_\pm, \mathbf{k}_\pm) = \Omega_s \pm \Omega_0$, tend to have $|\omega_\pm| \simeq |\omega_0|$ and $|\mathbf{k}_\pm| \simeq |\mathbf{k}_s|$, and may be regarded as short-wavelength (high-n) KAWs. $\Omega_\pm$, in turn, can interact with Ω_0; resulting in the nonlinear modification of eDW dispersion relation and, thereby, of its stability properties. The two-step scattering processes are illustrated schematically in figure 1. The first-step scattering process, i.e. the nonlinear generation of KAW sidebands is analyzed in the following section 3. We note here that, the analysis on the nonlinear generation of upper/lower KAW sidebands, is the same as that presented in [15], where KAW sidebands are generated as the test TAE scatters off ambient eDWs. The analysis is kept here for the readers' convenience and completeness. Meanwhile, section 4 analyzes the second step scattering process and the resultant nonlinear eDW dispersion relation.

3. Nonlinear generation of upper and lower sidebands of kinetic Alfvén waves

Let us first analyze the nonlinear generation of Ω_+; i.e, UKAW. The analysis for LKAW is similar. For electrons, we let $\delta g_{ke} = \delta g_{ke}^{(1)} + \delta g_{ke}^{(2)}$, with superscripts '(1)' and '(2)' denoting, respectively, the linear and nonlinear responses. Thus, from equation (2), we have

$$\delta g_{ke}^{(1)} \simeq -\frac{e}{T_e} F_{Me} \left(1 - \frac{\omega_{*e}}{\omega}\right)_k \delta\psi_k, \quad (7)$$

where $\delta\psi_k = (\omega \delta A_\|/ck_\|)_k$ is the effective potential due to the induced parallel electric field, $-\partial_t \delta A_\|/c$, and we have taken $|k_\perp \rho_e| \ll 1$ and the massless-electron $|\omega_k/k_\| v_{te}| \ll 1$ limit, with v_{tj} the thermal speed of the j-specie. In equation (7), k stands for the TAE/KAW modes; viz., Ω_0 and $\Omega_\pm$, and $\delta g_{se}^{(1)} \simeq 0$ as Ω_s is the predominantly electrostatic eDW mode. It then follows

$$\delta g_{+e}^{(2)} \simeq 0. \quad (8)$$

Meanwhile, for singly charged ions with $|\omega_k/k_\| v_{ti}| \gg 1$ for all the modes considered here, TAE, KAW and eDW, we have

$$\delta g_{ki}^{(1)} \simeq \frac{e}{T_i} F_{Mi} J_k \delta\phi_k \left(1 - \frac{\omega_{*i}}{\omega}\right)_k, \quad (9)$$

and

$$\delta g_{+i}^{(2)} \simeq -i \frac{\Lambda_0^s}{2\omega_+} J_0 J_s \frac{e}{T_i} F_{Mi} \left(\frac{\omega_{*i}}{\omega}\right)_s \delta\phi_s \delta\phi_0. \quad (10)$$

Substituting equations (7)–(10) into the quasi-neutrality condition, equation (5), it is possible to derive

$$\delta\psi_+ = \sigma_{*+} \delta\phi_+ + i \frac{\Lambda_0^s}{2\omega_+} D_+ \delta\phi_0 \delta\phi_s, \quad (11)$$

where

$$\sigma_{*+} = \left[1 + \tau - \tau \Gamma_+ (1 - \omega_{*i}/\omega)_+\right]/(1 - \omega_{*e}/\omega)_+, \quad (12)$$

and

$$D_+ = \tau(\omega_{*i}/\omega)_s F_+/(1 - \omega_{*e}/\omega)_+, \quad (13)$$

$\tau = T_e/T_i$, and $F_+ = \langle J_0 J_+ J_s F_{Mi} \rangle_v/N_0$. Meanwhile, the nonlinear gyrokinetic vorticity equation, equation (6), yields

$$\tau b_+ \left[\left(1 - \frac{\omega_{*i}}{\omega}\right)_+ \frac{1 - \Gamma_+}{b_+} \delta\phi_+ - \left(\frac{V_A^2 k_\| b k_\|}{b\omega^2}\right)_+ \delta\psi_+\right]$$
$$= -i \frac{\Lambda_0^s}{2\omega_+} \gamma_+ \delta\phi_s \delta\phi_0, \quad (14)$$

where

$$\gamma_+ = \tau \left[\Gamma_s - \Gamma_0 + (\omega_{*i}/\omega)_s (F_+ - \Gamma_s)\right]. \quad (15)$$

We note that, $k_\|$ and $b \propto k_\perp^2$ should be strictly considered as operators, and are thus, not commutative in, e.g. the field line bending term in equation (14). Combining equations (11)

and (14) then yields the equation describing the nonlinear generation of Ω_+ by Ω_0 and Ω_s; i.e.

$$\tau b_+ \epsilon_{A+}\delta\phi_+ = -\mathrm{i}(\Lambda_0^s/2\omega_+)\beta_+\delta\phi_s\delta\phi_0, \qquad (16)$$

where

$$\epsilon_{Ak} = \left(1 - \frac{\omega_{*i}}{\omega}\right)_k \frac{1-\Gamma_k}{b_k} - \left(\frac{V_A^2}{b}\frac{k_\parallel bk_\parallel}{\omega^2}\right)_k \sigma_{*k} \qquad (17)$$

is the linear SAW/KAW operator, and

$$\beta_+ = \tau(\Gamma_s - \Gamma_0) + \tau\left(\frac{\omega_{*i}}{\omega}\right)_s$$
$$\times \left[F_+ - \Gamma_s - \left(\frac{k_\parallel bk_\parallel}{\omega^2}\right)_+ \frac{\tau V_A^2 F_+}{(1-\omega_{*e}/\omega)_+}\right]. \qquad (18)$$

We note here that, on the right hand side of equation (16), ω_+ is kept consistently in the nonlinear term, which will finally enter R_+ defined in equation (42) that affects the stability of the test eDW. This is crucial especially when the present analysis is generalised to cases without clear frequency separation between the test DW and the ambient Alfvén instabilities, since as will be shown later that, the contribution of upper and lower KAW sidebands have opposite sign. Taking $|\omega_\pm| \simeq |\omega_0|$ in the KAW equations in [15], however, is qualitatively and quantitatively correct, since $|\omega_s| \ll |\omega_0|$, and the contribution of upper and lower KAW sidebands are additive.

Nonlinear generation of Ω_- follows that of Ω_+, and we, therefore, present only the main results. For electrons, we have, again, $\delta g_{-e}^{(2)} \simeq 0$, and, for ions,

$$\delta g_{-i}^{(2)} \simeq \mathrm{i}\frac{\Lambda_0^s}{2\omega_-}J_0 J_s \frac{e}{T_i}F_{Mi}\left(\frac{\omega_{*i}}{\omega}\right)_s \delta\phi_s\delta\phi_0^*. \qquad (19)$$

The quasi-neutrality condition, equation (5), yields,

$$\delta\psi_- = \sigma_{*-}\delta\phi_- - \mathrm{i}(\Lambda_0^s/2\omega_-)D_-\delta\phi_s\delta\phi_0^*, \qquad (20)$$

with

$$D_- = \tau(\omega_{*i}/\omega)_s F_-/(1-\omega_{*e}/\omega)_-, \qquad (21)$$

and $F_- = \langle J_0 J_- J_s F_{Mi}\rangle_v/N_0$. Meanwhile, the nonlinear gyrokinetic vorticity equation, equation (6), yields

$$\tau b_-\left[\left(1 - \frac{\omega_{*i}}{\omega}\right)_- \frac{(1-\Gamma_-)}{b_-}\delta\phi_- - \left(\frac{V_A^2 k_\parallel bk_\parallel}{b\omega^2}\right)_- \delta\psi_-\right]$$
$$= \mathrm{i}\frac{\Lambda_0^s}{2\omega_-}\gamma_-\delta\phi_s\delta\phi_0^*, \qquad (22)$$

and

$$\gamma_- = \tau\left[\Gamma_s - \Gamma_0 + (\omega_{*i}/\omega)_s(F_- - \Gamma_s)\right]. \qquad (23)$$

Finally, from equations (20) and (22), we have

$$\tau b_- \epsilon_{A-}\delta\phi_- = \mathrm{i}(\Lambda_0^s/2\omega_-)\beta_-\delta\phi_s\delta\phi_0^*. \qquad (24)$$

and

$$\beta_- = \tau(\Gamma_s - \Gamma_0) + \tau\left(\frac{\omega_{*i}}{\omega}\right)_s$$
$$\times \left[F_- - \Gamma_s - \left(\frac{k_\parallel bk_\parallel}{\omega^2}\right)_- \frac{\tau V_A^2 F_-}{(1-\omega_{*e}/\omega)_-}\right]. \qquad (25)$$

We remark, again, that $\epsilon_{A\pm}$ in equations (16) and (24) are KAW operators. That is, in terms of physics, equations (16) and (24) describe mode-converted KAWs ($\Omega_\pm$) driven by the nonlinear coupling between a TAE (Ω_0) and eDW (Ω_s).

4. Nonlinear dispersion relation of eDW

We now analyze the second scattering process between $\Omega_\pm$ and Ω_0 back into Ω_s. Again, let us first consider the Ω_+ channel; i.e. $\Omega_+ - \Omega_0 \to \Omega_s$. From the nonlinear gyrokinetic equation, equation (2), we have, for electrons in the massless $|\omega_k/k_\parallel v_{te}| \ll 1$ limit and noting equations (7) and (8),

$$\delta g_{se,+}^{(2)} \simeq -\mathrm{i}\frac{\Lambda_0^s}{2\omega_+}\frac{e}{T_e}F_{Me}\delta\psi_+\delta\psi_0^*\left[1 + \frac{k_{\parallel 0}}{k_{\parallel s}}\frac{(\omega_{*e}-\omega)_s}{\omega_0}\right]. \qquad (26)$$

Here, $\delta g_{se,+}^{(2)}$ denotes nonlinear electron response of Ω_s due to Ω_+ and Ω_0^* coupling. For ions, meanwhile, we have

$$\delta g_{si,+}^{(2)} \simeq \mathrm{i}(\Lambda_0^s/2\omega_s)\left(J_+\delta\phi_+\delta g_{0i}^{(1)*} - J_0\delta\phi_0^*\delta g_{+i}\right). \qquad (27)$$

Here, we note that $\delta g_{+i} = \delta g_{+i}^{(1)} + \delta g_{+i}^{(2)}$ is given, respectively, by equations (9) and (10). $\delta g_{si,+}^{(2)}$ is then given by

$$\delta g_{si,+}^{(2)} \simeq \left[\mathrm{i}\frac{\Lambda_0^s}{2\omega_+}J_0 J_+\delta\phi_+\delta\phi_0^* - \frac{(\Lambda_0^s)^2}{4\omega_s\omega_+}J_0^2 J_s|\delta\phi_0|^2\delta\phi_s\right]$$
$$\times \left(\frac{\omega_{*i}}{\omega}\right)_s \frac{e}{T_i}F_{Mi}. \qquad (28)$$

The analysis is similar for the $\Omega_- + \Omega_0 \to \Omega_s$ scattering channel. Then, we have

$$\delta g_{se,-}^{(2)} \simeq \mathrm{i}\frac{\Lambda_0^s}{2\omega_-}\frac{e}{T_e}F_{Me}\delta\psi_-\delta\psi_0\left[1 + \frac{k_{\parallel 0}}{k_{\parallel s}}\frac{(\omega_{*e}-\omega)_s}{\omega_0}\right], \qquad (29)$$

and

$$\delta g_{si,-}^{(2)} \simeq -\left[\mathrm{i}\frac{\Lambda_0^s}{2\omega_-}J_0 J_-\delta\phi_-\delta\phi_0 + \frac{(\Lambda_0^s)^2}{4\omega_s\omega_-}J_0^2 J_s|\delta\phi_0|^2\delta\phi_s\right]$$
$$\times \left(\frac{\omega_{*i}}{\omega}\right)_s \frac{e}{T_i}F_{Mi}. \qquad (30)$$

Substituting the $\delta g_{sj} = \delta g_{sj}^{(1)} + \delta g_{sj,+}^{(2)} + \delta g_{sj,-}^{(2)}$ for $j=e,i$ into the quasi-neutrality condition, equation (5), of the Ω_s mode, we then readily derive the following governing equation for $\delta\phi_s$;

$$\epsilon_s\delta\phi_s = \mathrm{i}(\Lambda_0^s/2\omega_+)\beta_s^+\delta\phi_0^*\delta\phi_+ - \mathrm{i}(\Lambda_0^s/2\omega_-)\beta_s^-\delta\phi_0\delta\phi_-$$
$$- \epsilon_s^{(2)}|\delta\phi_0|^2\delta\phi_s. \qquad (31)$$

Here, ε_s is the eDW linear dielectric operator and, in the limit of adiabatic circulating electrons and neglecting trapped electrons, is given by

$$\epsilon_s = 1 + \tau - \tau \left\langle \left(\frac{\omega - \omega_{*i}}{\omega - k_\| v_\| - \omega_d} \right)_s \frac{F_{Mi}}{N_0} J_s^2 \right\rangle_v ; \quad (32)$$

and, in the lowest order,

$$\epsilon_s \simeq 1 + \tau(1 - \Gamma_s) + \tau\Gamma_s(\omega_{*i}/\omega)_s. \quad (33)$$

Meanwhile,

$$\beta_s^\pm = \tau \left(\frac{\omega_{*i}}{\omega} \right)_s F_\pm + \sigma_{*0}\sigma_{*\pm} \left[1 + \frac{k_{\|0}}{k_{\|\pm}} \frac{(\omega_{*e} - \omega)_s}{\omega_0} \right], \quad (34)$$

and

$$\epsilon_s^{(2)} = \sum_{l=+,-} \left\{ \frac{F_2}{\omega_s\omega_l} + \sigma_{*0} \left[1 + \frac{k_{\|0}}{k_{\|l}} \frac{(\omega_{*e} - \omega)_s}{\omega_0} \right] \right.$$
$$\left. \times \left[\frac{F_l}{\omega_l^2(1 - \omega_{*e}/\omega)_l} \right] \right\} \frac{(\Lambda_0^s)^2}{4} \tau \left(\frac{\omega_{*i}}{\omega} \right)_s. \quad (35)$$

Noting equations (16) and (24) for, respectively, $\delta\phi_+$ and $\delta\phi_-$, equation (31) can be formally expressed as

$$\left(\epsilon_s + \epsilon_s^{(2)}|\delta\phi_0|^2 \right) \delta\phi_s = \left[\left(\frac{\Lambda_0^s}{2\omega_+} \right)^2 \frac{\beta_s^+ \delta\phi_0^* \beta_+}{\tau b_+ \epsilon_{A+}} \delta\phi_0 \right.$$
$$\left. + \left(\frac{\Lambda_0^s}{2\omega_-} \right)^2 \frac{\beta_s^- \delta\phi_0 \beta_-}{\tau b_- \epsilon_{A-}} \delta\phi_0^* \right] \delta\phi_s; \quad (36)$$

which may be regarded as the nonlinear eigenmode equation of Ω_s (eDW) in the presence of finite-amplitude Ω_0 (TAE) fluctuations, with $\epsilon_s^{(2)}$ from the diagonal nonlinear term, and the two terms on the right hand side from Ω_+ and Ω_- channels, respectively. Equation (36) has an interesting one-to-one correspondence with equation (24) of [15] on stability of TAE due to ambient eDW scattering.

Equation (36), in general, needs to be solved numerically. We can, however, make analytical progress by employing the scale separation and obtain an analytical dispersion relation variationally. First, we adopt the ballooning-mode representation for $\delta\phi_s$;

$$\delta\phi_s = \exp(\text{i} n_s \xi) \sum_{m_s} \exp(-\text{i} m_s \theta) \Phi_s(n_s q - m_s \equiv z_s), \quad (37)$$

where ξ and θ are, respectively the toroidal and poloidal angles, and denote the spatial scales of TAE and eDW as, respectively, $\mathbf{x}_0$ and $\mathbf{x}_s$; such that $|\mathbf{x}_s'|/|\mathbf{x}_0| \sim O(n_0/n_s) \ll 1$. Multiplying equation (36) by $\delta\phi_s^*$ and integrating over $\mathbf{x}_s$, we can derive

$$D_s + \chi_s^{(2)}|\delta\phi_0(\mathbf{x}_0)|^2 = R_+ + R_-, \quad (38)$$

where

$$D_s = \langle \Phi_s^*(z_s)\epsilon_s \Phi_s \rangle_s \quad (39)$$

is the linear dielectric constant of Ω_s,

$$\langle \Phi_s^*[A]\Phi_s \rangle_s \equiv \int_{-1/2}^{1/2} dz_s \sum_{m_s} \Phi_s^*[A]\Phi_s$$
$$= \int_{-\infty}^{\infty} dz_s \Phi_s^*[A]\Phi_s \quad (40)$$

with the normalization $\langle |\Phi_s|^2 \rangle_s = 1$,

$$\chi_s^{(2)} = \langle \Phi_s^* \epsilon_s^{(2)} \Phi_s \rangle_s, \quad (41)$$

and

$$R_\pm = \left\langle \Phi_s^* \left(\frac{\Lambda_s^s}{2\omega_\pm} \right)^2 \beta_s^\pm \left\{ \begin{matrix} \delta\phi_0^* \\ \delta\phi_0 \end{matrix} \right\} \frac{\beta_\pm}{(\tau b \epsilon_A)_\pm} \left\{ \begin{matrix} \delta\phi_0 \\ \delta\phi_0^* \end{matrix} \right\} \Phi_s \right\rangle_s. \quad (42)$$

Equation (38) is formally the variational nonlinear eDW dispersion relation in the presence of a finite-amplitude TAE given by $\delta\phi_0$. We will later analyze it further using a trial function for $\Phi_s(z_s)$. We now make some qualitative observations. We note that $\chi_s^{(2)}$ is real and, in general, $R_\pm = Re(R_\pm) + \text{i} Im(R_\pm)$. Thus, $\chi_s^{(2)}$ and $Re(R_\pm)$ lead to nonlinear frequency shift; while $Im(R_\pm)$ gives rise to nonlinear damping or growth. Focusing on $Im(R_\pm)$ first, we observe, from equation (42), that $Im(R_\pm) \propto Im(1/\epsilon_{A\pm})$; i.e. the imaginary component of the SAW/KAW operator, $\epsilon_{A\pm}$, given by equation (17). Looking at equation (16) and letting

$$\delta\phi_+ = A_+(\mathbf{x}_0)\exp(\text{i} n_s \xi)$$
$$\times \sum_{m_s} \exp(-\text{i} m_s \theta)\Phi_+(z_s \equiv n_s q - m_s), \quad (43)$$

we then have, recalling the scale separation between $\mathbf{x}_0$ and $\mathbf{x}_s$,

$$A_+(\mathbf{x}_0)\tau b_s \epsilon_{A+}^s \Phi_+(z_s) = -\text{i} \frac{\Lambda_0^s}{2\omega_+} \beta_+ \Phi_s(z_s)\delta\phi_0(\mathbf{x}_0). \quad (44)$$

The same analysis can be carried out for $\delta\phi_-$ given by equation (24) step by step. Further simplification of equation (44) and the analogue for the Ω_- sideband can be obtained noting that

$$\epsilon_{A\pm}^s = \left(1 - \frac{\omega_*}{\omega} \right)_\pm \frac{1 - \Gamma_s}{b_s} - \left(\frac{V_A^2}{b_s} \frac{k_\| s b_s k_\| s}{\omega_\pm^2} \right) \sigma_{*\pm}^s,$$
$$\sigma_{*\pm}^s \simeq \left[1 + \tau - \tau\Gamma_s(1 - \omega_{*i}/\omega)_\pm \right] / (1 - \omega_{*e}/\omega)_\pm, \quad (45)$$

$b_s = b_{s\theta}(1 + \hat{s}^2\partial_{z_s}^2)$, $b_{s\theta} = k_{s\theta}^2\rho_i^2$, $\hat{s} = rq'/q$ denotes magnetic shear, and $k_{\|s} = (n_s q - m_s)/(qR) = z_s/(qR)$. Since $|\hat{s}^2\partial_{z_s}^2| < 1$ for moderately/strongly ballooning modes, taking Taylor expansion of $\epsilon_{A\pm}^s$ at $b_s = b_{s\theta} + b_{s\theta}\hat{s}^2\partial_{z_s}^2$, one obtains

$$\epsilon_{A\pm}^s \simeq b_{s\theta} \frac{\partial \epsilon_{A\pm}^s}{\partial b_{s\theta}} \hat{s}^2\partial_{z_s}^2 - \left(\frac{\omega_A}{\omega_\pm} \right)^2 \sigma_{\pm s}(z_s^2 - z_\pm^2). \quad (46)$$

Here, $\omega_A = V_A/(qR)$,

$$\sigma_{\pm s} = [1 + \tau - \tau \Gamma_s(b_{s\theta})(1 - \omega_{*i}/\omega)_\pm]/(1 - \omega_{*e}/\omega)_\pm, \tag{47}$$

$z_s^2 = k_{\parallel s}^2 q^2 R^2$, and

$$z_\pm^2 = \left(\frac{\omega}{\omega_A}\right)_\pm^2 \left(1 - \frac{\omega_{*i}}{\omega}\right)_\pm \frac{1 - \Gamma_s(b_{s\theta})}{b_{s\theta}} \frac{1}{\sigma_{\pm s}} < \frac{1}{4}; \tag{48}$$

as $|\omega/\omega_A|_\pm^2 \simeq 1/4$. Equation (44), along with ϵ_{A+}^s given by equation (46), indicates that the upper sideband is a mode converted KAW at the high-n Alfvén resonance layer $z_s = \pm z_+$. As noted in previous study of mode-converted KAW [16], for $\tau = T_e/T_i \sim 1$, the finite electron Landau damping as well as the Airy swelling of the amplitude dictate that the damping occur predominantly around $z = \pm z_+$. Furthermore, the spectrum of eDW is typically broad, which implies that the spectrum of mode-converted KAW is correspondingly broad. Thus, the KAW energy absorption rate approximates that of the local Alfvén resonance via the causality constraint $Im(\omega_s, \omega_+, \omega_-) > 0$, and may be estimated by the Alfvén resonance absorption in the following analysis [21–23]; i.e.

$$Im\left(\frac{1}{\epsilon_{A+}}\right) \simeq -\pi\delta(\epsilon_{A+}) \simeq -\pi\left(\frac{\omega_+}{\omega_A}\right)^2 \frac{\delta(z_s^2 - z_+^2)}{\sigma_{+s}}. \tag{49}$$

Similar processes occur for the Ω_- KAW; i.e.

$$Im\left(\frac{1}{\epsilon_{A-}}\right) \simeq \pi\delta(\epsilon_{A-}) \simeq \pi\left(\frac{\omega_-}{\omega_A}\right)^2 \frac{\delta(z_s^2 - z_-^2)}{\sigma_{-s}}. \tag{50}$$

Consequently,

$$Im(R_+) = -\left\langle \Phi_s^*\left(\frac{\Lambda_0^s}{2\omega_+}\right)^2 \frac{\beta_s^+ \delta\phi_0^* \beta_+}{\tau b_s} \right. $$
$$\left. \times \left(\frac{\omega_+}{\omega_A}\right)^2 \frac{\delta(z_s^2 - z_+^2)}{\sigma_{+s}} \delta\phi_0 \Phi_s \right\rangle, \tag{51}$$
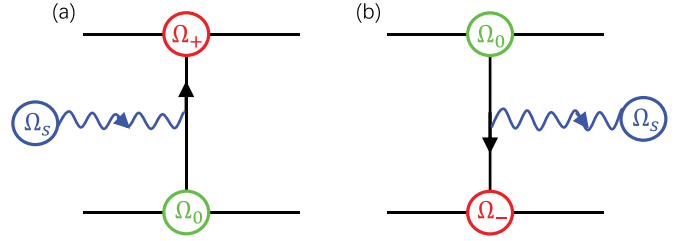
and, omitted here, a similar corresponding expression can be obtained for $Im(R_-)$.

To proceed further, we take the $|k_{0\perp}\rho_i|^2 \ll 1$ limit but keep the finite $|\omega_s/\omega_0| < 1$ correction. It is then straightforward to derive

$$\beta_s^\pm \beta_\pm \delta(\epsilon_{A\pm}) \simeq \tau(1 - \Gamma_{s\theta})\sigma_{s\theta} \frac{(\omega_{*e} - \omega)_s(\omega - \omega_{*i})_s}{\omega_0^2}$$
$$\times \frac{k_{\parallel\pm}}{k_{\parallel s}}\delta(\epsilon_{A\pm}). \tag{52}$$

Here, $\Gamma_{s\theta} = \Gamma_s(b_{s\theta})$ and $\sigma_{s\theta} = 1 + \tau(1 - \Gamma_{s\theta})$.

The anti-Hermitian part of the variational nonlinear eDW dispersion relation, equation (38), then yields, letting $\omega_s = \omega_{sr} + i\gamma_s$ and $D_{sr}(\omega_{sr}) = 0$,

Figure 2. Sketch illustrating the nonlinear scattering processes of (*a*) stimulated absorption and (*b*) spontaneous emission. Here, Ω_0 is the finite-amplitude TAE, Ω_s is the test eDW, and $\Omega_\pm$ are, respectively, the upper and lower sideband KAWs.

$$\left(\gamma_s + \gamma_s^l\right)\frac{\partial}{\partial\omega_{sr}}D_{sr} = Im(R_+ + R_-)$$
$$\simeq -\frac{\pi}{4\beta_i}\left(\frac{\Omega_i}{\omega_0}\right)^2\left|\frac{\delta B_{0\theta}}{B_0}\right|^2(1 - \Gamma_{s\theta})\sigma_{s\theta}$$
$$\times \frac{(\omega_{*e} - \omega)_s(\omega - \omega_{*i})_s}{\omega_0^2}\left\langle\left[\left(\frac{\omega_+}{\omega_A}\right)^2\frac{\delta(z_s^2 - z_+^2)}{\sigma_{+s}}\right.\right.$$
$$\left.\left. -\left(\frac{\omega_-}{\omega_A}\right)^2\frac{\delta(z_s^2 - z_-^2)}{\sigma_{-s}}\right]|\Phi_s|^2\right\rangle_s. \tag{53}$$

Here, γ_s is the operator for slowly temporal variation of Ω_s amplitude, $\gamma_s^l \equiv Im(D_s)/(\partial D_{sr}/\partial\omega_{sr})$ is the linear damping/growth of eDW. Noting that $\partial D_{sr}/\partial\omega_{sr} > 0$, $Im(R_+) < 0$ and $Im(R_-) > 0$, scatterings to UKAW and LKAW, thus, lead to, respectively, damping and growth of eDW. As illustrated in figure 2, one may qualitatively regard UKAW scattering as stimulated absorption, and LKAW scattering as spontaneous emission, similar to the familiar parametric decay instability via a quasi-mode [24]. Since the rate of the two processes are nearly the same, the number of Ω_s quanta are approximately conserved, with a small overall effect on eDW growth/damping. In equation (53), we have also noted $ck_{0r}\delta\phi_0/B_0 \simeq V_A\delta B_{0\theta}/B_0$.

To estimate the nonlinear damping/growth rate quantitatively, we adopt a trial function for Φ_s as $|\Phi_s|^2 = (1/\sqrt{\pi}\Delta_s)\exp(-z_s^2/\Delta_s^2)$ with $\Delta_s > 1$ for a typical moderately ballooning eDW. Equation (53) then yields

$$Im(R_+ + R_-) \simeq -\frac{\sqrt{\pi}}{4\beta_i}\left(\frac{\Omega_i}{\omega_0}\right)^2\left|\frac{\delta B_{0\theta}}{B_0}\right|^2(1 - \Gamma_{s\theta})\sigma_{s\theta}$$
$$\times \frac{(\omega_{*e} - \omega)_s(\omega - \omega_{*i})_s}{\omega_0^2}$$
$$\times \left[\left(\frac{\omega_+}{\omega_A}\right)^2\frac{1}{\sigma_{+s}z_+\Delta_s} - \left(\frac{\omega_-}{\omega_A}\right)^2\frac{1}{\sigma_{-s}z_-\Delta_s}\right]. \tag{54}$$

Taking typical tokamak parameters, $\Omega_i/\omega_0 \sim O(10^2)$, $\beta_i \sim O(10^{-2})$, $b_{s\theta} \sim O(1)$, $|\omega_s/\omega_0|^2 \sim O(10^{-1})$, and $|\Delta_s z_\pm| \sim O(1)$, we then find

$$|Im(R_+ + R_-)| < O(10^5)\left|\frac{\delta B_{0\theta}}{B_0}\right|^2. \tag{55}$$

Noting that $\partial D_{sr}/\partial \omega_{sr} \sim 1/\omega_{sr}$ and $\gamma_s^l/\omega_{sr} \sim O(10^{-1})$ as, e.g. in the trapped electron mode [25], we then find that, for TAE fluctuations with $|\delta B_{0\theta}/B_0|^2 \lesssim O(10^{-7})$ [26], the nonlinear contribution of damping/growth $\sim |Im(R_+ + R_-)| \lesssim O(10^{-2}) \ll |\gamma_s^l/\omega_s|$, and should have negligible effects on the eDW stability. We also remark that one can, furthermore, straightforwardly show that the nonlinear frequency shift due to $\chi_s^{(2)}|\delta\phi_0|^2$ and $Re(R_+ + R_-)$ is also typically negligible.

5. Conclusions and discussions

In this work, we have employed the nonlinear gyrokinetic equations and investigated analytically direct wave–wave interactions between a test eDW and ambient finite-amplitude TAEs in low-β circular tokamak plasmas. Here, nonlinear scatterings generate upper and lower sidebands of mode-converted KAWs at high toroidal mode numbers which are typically damped by electrons around the mode conversion positions. Furthermore, we find that scattering to upper-sideband KAW gives rise to stimulated absorption and, hence, damping of the eDW. Scattering to lower-sideband KAW, on the other hand, gives rise to spontaneous emission and, thereby, growth of the eDW; i.e. TAE parametrically decays to eDW via the lower-sideband KAW quasi-mode. For typical tokamak parameters and TAE fluctuation intensity, our analysis indicates that the net effects on eDW stability properties should be negligible. Note that while the present analysis is done for a single-n TAE in order to simplify the notations, the results and conclusions can obviously be extended to the case of multiple-n TAEs. We remark again that, as noted in section 1, the present results are different with those obtained previously for the case of direct wave–wave interactions between a test TAE and ambient eDW [15]. In that case, both channels of scatterings to KAWs lead to stimulated absorption and, thereby, significant damping of the TAE.

As noted above, our analysis adopts the electron drift waves without temperature gradients as a paradigm model in order to simplify the analysis and delineate more clearly the underlying nonlinear physics mechanisms. It is clearly desirable to extend the investigations to include ITG modes, trapped particle effects, as well as other types of AEs; such as reversed shear Alfvén eigenmodes (RSAEs) [27, 28] and beta-induced Alfvén eigenmodes (BAEs) [29, 30]. While detailed analyses for such cases remain to be carried out, one may conjecture that the physical pictures outlined in the current paradigm model should hold at least qualitatively. This fundamental difference in the underlying physics processes is the reason why TAE scattering off ambient eDW results into significant enhancement of TAE damping [15], while eDW scattering off ambient TAE yields negligible impact on eDW stability, as shown in the present analysis. The spatial scale separation is the same in the two processes, which can be viewed as one being the reverse of the other. Furthermore, the scattered waves are always high frequency KAWs in both processes. However, since the lower-sideband KAW has lower frequency than ambient TAE, the corresponding nonlinear coupling results into spontaneous emission rather than stimulated absorption.

Finally, that the present results indicating negligible effects on eDW via direct wave–wave interactions with TAE suggests the possible significance of indirect interaction via, e.g. the zonal structures consisting of flow, field and phase space components nonlinearly generated by AEs [8, 10, 31, 32]. The present results seem to suggest that the indirect interaction channel should play a dominant role when AEs and DWs are characterized by a clear frequency separation. However, for low frequency drift Alfvén waves, frequency separation with DWs and, thus, the predominance of the indirect coupling channel is less obvious. This interesting subject remains to be further investigated in the future.

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