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Non-inertial physics of action-angle variables of non-trivial periodic motion

Francesco Vergari[1] **and Michele Marrocco**[2,*]

[1] Dipartimento di Fisica, Università di Roma 'La Sapienza', Rome I-00185, Italy
[2] ENEA (Italian National Agency for New Technologies, Energy and Sustainable Economic Development), via Anguillarese 301, Rome I-00123, Italy

E-mail: michele.marrocco@enea.it

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Abstract

Action-angle (AA) variables provide a framework to study periodic motion with the basic tools of Hamiltonian mechanics. They require a reference frame for their definition as physical observables (angles and angular momenta) in real space, but the nature of this frame is usually disregarded. We employ the *active* perspective of classical mechanics to demonstrate that the AA approach introduces a mapping between non-trivial periodic motion (i.e. when instantaneous frequencies deviate from the system's characteristic frequencies) observed in an inertial frame and simple circular trajectories observed in a suitably chosen non-inertial frame where the AA variables can be defined as physical observables. After a general description of the critical aspects in the AA approach to non-trivial periodic motion, the link between AA variables and non-inertial frames is illustrated through the pedagogically rich example of the two-dimensional isotropic harmonic oscillator, a model representative of the broad class of Hamiltonians with non-trivial periodic signature. Our key conclusion is that the familiar representation of AA variables on tori in phase space is not merely a mathematical abstraction but corresponds to a topological transformation induced by the Euler force thanks to a change of reference frames in real space, from inertial to non-inertial. This work not only enhances students' understanding of AA variables but also provides a pedagogical opportunity to underline the role of the non-inertial Hamiltonian mechanics, often limited to few examples free from the Euler force.

* Author to whom any correspondence should be addressed.

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1. Introduction

Hamilton's equations of motion are frame-independent provided that the transformed Hamiltonian in the accelerated frame incorporates the appropriate energy terms accounting for inertial effects [1, 2]. Interestingly, the same transformation rule apply in quantum mechanics as well [3, 4]. For example, in rotating reference frames, both classical and quantum Hamiltonians can be directly related to their inertial counterparts, as long as the inertial Hamiltonian is re-expressed in terms of the non-inertial variables and added to the proper correction term dependent on the angular momentum [1–4].

This important feature of Hamiltonian mechanics is not discussed in the context of canonical transformations leading to the definition of action-angle (AA) variables. To address this gap, we aim to demonstrate how common AA variables are intrinsically tied to the description of the periodic motion in accelerated frames.

The AA approach is an outgrowth of the Hamilton–Jacobi (HJ) theory of classical mechanics [2, 5]. It aims at finding the frequencies of a periodic motion without paying too much attention to the details of the motion itself. Students learn about the AA variables in classes of analytical mechanics, but the approach is rediscovered when rudiments of the so-called old quantum theory are introduced (e.g. Bohr–Sommerfeld quantization rule and Sommerfeld's solution to the energy levels of the hydrogen atom [6–9]). Note, however, that the quantum interpretation of AA variables remains an active area of research [10–20]. For this reason, the conclusions of this study may also have some relevance for the quantum approach to AA variables.

In teaching classical AA theory, the passive view of canonical transformations is usually employed [5]. This perspective involves redefining the phase-space mapping, resulting in a reformulation of the Hamiltonian as the dynamical description undergoes the change from the original coordinates and momenta to the AA variables. Within this framework, the calculation of the action variables plays the primary role and examples are available in textbooks [2, 5, 21, 22] and other resources [23–29]. The relevance of the action variables has an easy explanation. They are constants of the periodic motion and, when treated as independent variables, allow the system's frequencies to be determined.

Angle variables tend to receive comparatively little attention in educational materials. Consequently, their physical significance as observables within a specific reference frame is rarely explored in depth. This oversight leaves unresolved a fundamental question: which reference frame is associated with the full set of AA variables?

In addressing the question, this work advocates for greater attention to phase variables, meaning both angle coordinates and angle variables. The transformation between them suggests a reinterpretation of the AA theory within the active view of Hamiltonian mechanics. Differently from the passive view, the active reinterpretation involves modifying the dynamical state while maintaining the phase space mapping [5].

A notable application of the active view emerges in the AA treatment of non-trivial periodic motion, which occurs when the instantaneous frequencies (in the inertial frame) deviate, almost continuously, from the constant characteristic frequencies of the underlying periodic motion. Prominent examples of educational value include the isotropic harmonic oscillator and Kepler's problem. In such cases, actively transforming the ignorable angle into its corresponding angle variable reveals the shift from the inertial framework of the original HJ problem to a non-inertial reference frame, whose motion adds a phase to the ignorable angle, thereby facilitating the extraction of constant frequencies through the AA formalism.

To illustrate the shift from the passive to the active view in a fashion suitable for students taking classes of classical mechanics, we use the two-dimensional (2D) isotropic oscillator. The insights gained from this paradigmatic example can then be applied to more complex systems. Our key conclusion is thus that a change of reference frames in real space, from inertial to non-inertial, motivates the representation of non-trivial periodic motion on tori in the phase space.

The work is organized as follows. Section 2 provides a brief summary of the AA variables under the passive view of classical mechanics. Section 3 describes critical aspects of non-trivial periodic motion. Section 4 and its subsections illustrate the non-inertial interpretation of the 2D isotropic oscillator within the context of the active view of classical mechanics. A final section 5 presents the conclusions summarizing the key findings and insights derived from the analysis.

2. Passive view of AA variables

The AA approach is explained in several texts [2, 5, 22]. Here, we summarize it for the Reader who is less familiar with the tools of AA theory. More experienced Readers can skip this section.

Let us introduce a periodic mechanical system whose Hamiltonian depends on a set $q = \{q_1, q_2,\dots,q_n\}$ of coordinates and on the corresponding set $p = \{p_1, p_2,\dots,p_n\}$ of momenta. The AA method applied to such a mechanical system relies on the role of the Hamilton's characteristic function W as generator of the canonical transformation $(q, p) \to (\phi, J)$, with ϕ and J indicating the two vectors $\phi = \{\phi_1, \phi_2,\dots,\phi_n\}$ and $J = \{J_1, J_2,\dots,J_n\}$. This transformation induces a new mapping of the phase space that determines a new form of the Hamiltonian without altering the dynamical state (passive view of canonical transformations [5, 22]). Since W is a type 2 generating function depending on the old coordinates q and the new momenta J, the transformation laws involve the following derivatives of W

$$J_i = \frac{1}{2\pi}\oint \frac{\partial W}{\partial q_i}\mathrm{d}q_i, \tag{1}$$

$$\phi_i = \frac{\partial W}{\partial J_i}. \tag{2}$$

The canonical momentum $p_i = \partial W/\partial q_i$ is integrated over a complete orbit, whereas the derivative $\partial W/\partial J_i$ can be potentially used to find the law of the motion $q = q(\phi, J, t)$. The definition in equation (1) is not unique. An equivalent definition can be introduced where the action variables are made of circuital integrations of the scalar product $\nabla W\cdot\mathrm{d}q$ [2]. More in general, it is possible to look for action variables that are taken as some constants of the motion that are in involution and satisfy the other assumptions of the Liouville integrability theorem [2, 22].

Independently from how the action variables are found, the beauty of the method comes from the corresponding Hamilton's equations $\dot{J}_i = 0$ and $\dot{\phi}_i = \omega_i(J)$ (the dots indicate time derivatives). They reveal linearly time-dependent angle variables whose time derivatives are the frequencies $\omega_i(J)$ of the motion. The knowledge of each pair (J_i, ϕ_i) facilitates the visualization of the motion on an n-dimensional torus $\mathbb{T}^n$ where the actions J_i label the radii of the torus and the angle variables mark the periodicity on the torus.

Much of the didactic success of the AA method is based on the calculation of the loop integrals of equation (1). The derivatives of equation (2) are, instead, disregarded. As a matter of fact, the angle variables are not used to find the frequencies of the mechanical system. They are replaced with the Hamilton's equations $\omega_i(J) = \partial H(J)/\partial J_i$, where $H(J)$ is the transformed Hamiltonian, and again the primary role of the action variables is reiterated. The bottom line is that the angle variables fade away into the background of the whole AA picture. In contrast with this scenario, the current work underlines the relevance of the phase information in understanding the AA approach to periodic motion featuring time-dependent instantaneous frequencies.

3. Critical aspects of non-trivial periodic motion

Non-trivial periodic motion can be introduced in contrast with trivial harmonic motion. This is characterized by a constant frequency, with classic examples including the mass-spring system, simple pendulum (in the small-angle approximation), and LC electronic circuits. These systems are governed by linear restoring forces, leading to simple, predictable periodic behavior.

However, beyond these idealized cases, most realistic periodic systems exhibit more complex dynamics. In particular, systems influenced by nonlinear forces or involving multiple degrees of freedom often display instantaneous frequencies that vary in time. While their motion remains periodic, it cannot be easily reduced to a set of angle variables that evolve linearly in time.

Interestingly, this difference can also arise in systems that are fundamentally harmonic in one reference frame and show anharmonic features for another choice of coordinates. A prime example is the 2D isotropic harmonic oscillator, which is trivially harmonic in Cartesian coordinates with constant frequencies along each axis. Yet, when described in polar coordinates, the system's angular evolution becomes nonlinear in time, leading to a non-uniform angular frequency. This occurs despite the underlying linearity of the restoring force.

This example challenges the traditional definition of anharmonic motion, as it demonstrates that apparent anharmonic features, such as time-varying frequencies, can emerge from coordinate transformations, even in systems with recognized harmonic dynamics.

For this reason, it is more accurate and conceptually useful to distinguish between trivial periodic motion and its non-trivial counterpart.

3.1. Formal definition of AA variables for trivial periodic motion

In section 2, we have introduced the AA variables (ϕ, J) for a completely integrable mechanical system which is periodic in n degrees of freedom. A key assumption in our argument is that the generalized coordinates $q = \{q_1, q_2, \ldots, q_n\}$ and conjugate momenta $p = \{p_1, p_2, \ldots, p_n\}$ are defined in an inertial reference frame. Under such conditions, one can perform a canonical transformation from the original phase-space variables (q, p) to the new variables (ϕ, J) that parametrize the torus in the transformed phase space. Now, suppose a point transformation exists in the configuration space such that the canonical coordinates can be directly related to angle coordinates (e.g. from Cartesian to polar or spherical coordinates). Since point transformations of this kind are inherently canonical [5], the full transformation $(q, p) \to (\phi, J)$ may proceed through an intermediate step where q_i is replaced by φ_i. In particular, the transformation may even include the identity $q_i = \varphi_i$, implying that the angle coordinate φ_i was chosen as the canonical coordinate q_i from the outset. Note that, if more than one angle coordinate is involved in the intermediate step, than the following definition can be verified for all the angles employed.

Let us then define *trivial periodic motion* as occurring when an angle coordinate φ_i evolves with instantaneous frequency $\dot{\varphi}_i$ that exactly matches one constant characteristic frequency $\dot{\phi}_i = \omega_i(J)$ of the periodic motion. The condition suggests that the trivial angle coordinate φ_i and the angle variable ϕ_i may differ only by a constant phase shift.

The simplest and most illustrative example of trivial periodic motion is that of a simple harmonic oscillator, which involves a single angle coordinate ($n = 1$). For clarity, we omit subscripts and denote the oscillator's characteristic frequency by ω, which coincides with the time derivative of the angle variable ϕ as defined by equation (2). On the other hand, the angle coordinate $\varphi = \omega t + \varphi_0$ appears in the law of motion $q = q_0 \sin(\varphi)$ which establishes the sought point transformation ruling the coordinate replacement $q \to \varphi$ [5]. The frequency $\dot{\varphi} = \omega$ thus satisfies the condition $\dot{\varphi} = \dot{\phi} = \omega$ requested for the definition of trivial periodic motion.

3.2. Formal definition of non-trivial periodic motion and consequences for the active view of AA variables

After introducing trivial periodic motion, we now turn to its counterpart: non-trivial periodic motion. Consider again the integrable periodic system introduced in section 2, which has n degrees of freedom and canonical variables (q, p) defined in an inertial reference frame. Due to the system's integrability, the Liouville-Arnold theorem ensures the existence of AA variables (ϕ, J) [22]. Suppose that one of these action variables, called J_i, is also conjugate to an angle coordinate φ_i taken in the inertial reference frame (recall that point transformations of the configuration space are always canonical [5]). In this context, we define non-trivial periodic motion for the ith degree of freedom if the condition $\dot{\varphi}_i \neq \dot{\phi}_i = \omega_i(J)$ holds. The inequality indicates that the non-trivial frequency $\dot{\varphi}_i$ varies with time because $\dot{\varphi}_i$ cannot be any constant apart from a system's characteristic frequency. As a result, the non-trivial angle coordinate φ_i evolves nonlinearly, in contrast to the linear time evolution of the corresponding angle variable ϕ_i.

In explaining the definition, we first establish that the only case in which $\dot{\varphi}_i$ can be constant is when it equals a characteristic frequency of the system. We begin with the generating function $f_i(q)J_i$ of the the point transformation leading to the angle coordinate $\varphi_i = f_i(q)$ conjugate to J_i ($f_i(q)$ is a function of the generalized coordinates q). Next, we suppose that $\dot{\varphi}_i = \lambda_i$, where λ_i is some constant with no *a priori* physical interpretation. From Hamilton's equations, $\dot{\varphi}_i = \partial H/\partial J_i$ implies $\dot{\varphi}_i = \lambda_i = \omega_i(J)$ because the derivative $\partial H/\partial J_i$ gives the characteristic frequency $\omega_i(J)$ of the periodic system [5, 22]. The conclusion is that if $\dot{\varphi}_i$ was constant, it would coincide with one characteristic frequency of the motion, and hence φ_i would follow the trivial motion defined before. That is, φ_i would coincide with the angle variable ϕ_i up to a constant phase shift. However, by assumption, φ_i is not an angle variable. The only remaining physically meaningful alternative is that $\dot{\varphi}_i$ is not constant. Consequently, φ_i must be a nonlinear function of time, confirming the presence of non-trivial periodic motion.

Despite this nonlinearity, the phases φ_i and ϕ_i can still be related through an appropriate transformation of the form

$$\phi_i = \varphi_i + \Theta_i, \tag{3}$$

where Θ_i accounts for and effectively removes the nonlinear behavior of φ_i. This transformation naturally arises from the definition of the angle variable given in equation (2), and it can be interpreted as a shift from an inertial reference frame to a new one undergoing curvilinear translational motion with angular frequency $\Omega_i = -\dot{\Theta}_i$ (the reason for the minus sign will become clear later). Note that the transformation of equation (3) is particularly easy to derive in mechanical systems that experience central forces resulting in cylindrical or spherical symmetry. In such cases, one or more angle coordinates φ_i are ignorable and equation (3) becomes evident under the light of the separability of the Hamilton's characteristic function.

Within the non-inertial interpretation of equation (3), the role of $\dot{\Theta}_i$ is clear: it compensates for the time-dependent variations in $\dot{\varphi}_i$, ensuring that $\dot{\phi}_i$ remains constant. In effect, the transformation to an accelerated frame absorbs the nonlinearity of the original motion in the inertial frame, thereby mapping a non-trivial periodic behavior into a trivial (or linear) one observed in the non-inertial frame. This highlights that the AA approach to non-trivial periodic motion is, in fact, a consequence of the transformation between reference frames, from inertial to non-inertial.

Finally, a key insight regarding equation (3) is that transitioning to a non-uniformly accelerated reference frame alters the apparent laws of motion due to the emergence of fictitious forces and their inertial effects. The strength of the AA approach lies in its ability to harness these effects, which are not merely a complication but are actually employed to reveal the underlying regularity of non-trivial periodic motion. In particular, the fictitious Euler force, arising from the time variation of the angular velocity of the accelerated frame, plays a central role here.

3.3. Non-inertial nature of non-trivial AA Hamiltonians

From the analysis developed so far about our n-dimensional system, we have learnt that the transformation $(q, p) \rightarrow (\phi, J)$ involves a subtle distinction. The set ϕ of angle variables can be split into two subsets. The subset $\phi_t = \{\phi_1, \phi_2, ..\phi_m\}$ of angle variables, corresponding to $m \leqslant n$ trivially periodic degrees of freedom, is indeed free from additional angles that are instead needed to form the subset $\phi_{nt} = \{\bar{\phi}_1, \bar{\phi}_2, ..\bar{\phi}_{n-m}\}$ of non-trivial angle variables arising from transformations akin to equation (3). Accordingly, the full transformation leading to the AA variables should be more accurately written as $(q, p) \rightarrow (\phi_t, \phi_{nt}, J_t, J_{nt})$, where $J_t = \{J_1, J_2, ..J_m\}$ and $J_{nt} = \{\bar{J}_1, \bar{J}_2, ..\bar{J}_{n-m}\}$ are the two conjugate subsets of action variables.

The separation between trivial and non-trivial AA variables sheds light on the non-inertial nature of ϕ_{nt}. This subset is nothing but $\phi_{nt} = \varphi_{nt} + \Theta$, where $\varphi_{nt} = \{\bar{\varphi}_1, \bar{\varphi}_2,\ldots,\bar{\varphi}_{n-m}\}$ denotes the subset of non-trivial angle coordinates in the inertial frame and $\Theta = \{\Theta_1, \Theta_2,\ldots,\Theta_{n-m}\}$ is the subset of auxilary angles responsible for the removal of the nonlinearities in φ_{nt}. In contrast, a simpler relationship exists between trivial angle variables in ϕ_t and their inertial counterparts in $\varphi_t = \{\varphi_1, \varphi_2,\ldots,\varphi_m\}$. Note that, due to their trivial nature, ϕ_t and φ_t are dynamically equivalent (i.e. same Hamilton's equation for ϕ_i and φ_i). At most, their corresponding components may differ by constant phase shifts, which can be disregarded or set to zero without loss of generality.

The practical consequence of this separation is the decomposition of the global transformation $(q, p) \rightarrow (\phi, J)$ into two distinct steps. The first takes place in the inertial reference frame and is

$(q, p) \rightarrow (\varphi_t, \varphi_{nt}, J_t, J_{nt})$. The second is $(\varphi_t, \varphi_{nt}, J_t, J_{nt}) \rightarrow (\phi_t, \phi_{nt}, J_t, J_{nt}) = (\phi, J)$ and takes place in the non-inertial frame where the cancellation of the nonlinearities in φ_{nt} occurs. Remarkably, the second step introduces the effect of the angles in Θ. Their influence is manifest in the Hamiltonian as we will proceed to show.

Consider the most general transformation between Hamiltonians resulting from the transformation between two sets of canonical variables related by the generating function F [2, 5]

$$K = H + \frac{\partial F}{\partial t}. \tag{4}$$

Suppose that K (otherwise known as Kamiltonian [5]) represents the Hamiltonian in terms of AA variables. In this case, K depends only on the conserved angular momenta (i.e. the action variables) while the angle variables are cyclic or ignorable. However, despite being ignorable in K, the angle variables and their non-trivial angle coordinates shape the transformation of equation (4). To demonstrate this, we introduce again the point transformation used in the explanation of the definition of non-trivial periodic motion (see section 3.2 for context). That transformation leads to the replacement in H of the original canonical coordinates with angle coordinates conjugate to the action variables of K. Now, consider the set φ_{nt} of non-trivial angle coordinates in the inertial Hamiltonian H. We have seen before that these angles are related to the analogous set ϕ_{nt} of ignorable angle variables of the Kamiltonian via $\phi_{nt} = \varphi_{nt} + \Theta$. Taking advantage of the Hamilton's equations, the time derivatives of the components $\overline{\phi}_i$, $\overline{\varphi}_i$ and Θ_i allow us to write $\partial K/\partial \overline{J}_i = \partial H/\partial \overline{J}_i - \Omega_i$, where we have introduced the frequencies $\Omega_i = -\dot{\Theta}_i$ (the reason for the minus sign will be clarified shortly). The equation can be integrated to yield the relationship between H and K

$$K = H - J_{nt} \cdot \Omega. \tag{5}$$

Here, $\Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_{n-m}\}$, and $J_{nt} \cdot \Omega$ denotes the dot product over the indices of the degrees of freedom presenting non-trivial periodic dynamics in the inertial reference frame. It is therefore easy to recognize that equation (5) is consistent with equation (4) provided the generating function F satisfies the condition $\partial F/\partial t = -J_{nt} \cdot \Omega$.

Equation (5) is a well-established result in classical mechanics describing how the Hamiltonian transforms in non-inertial reference frames [1, 2]. The same transformation also holds for relativistic quantum motion in rotating frames [30–32] and for wave Hamiltonians in optics [33]. In all such cases, the scalar product $J_{nt} \cdot \Omega$ encapsulates inertial effects that highlights the action of an accelarated frame necessary to suppress the nonlinearities of the non-trivial periodic motion. In conclusion, inertial effects are responsible for transforming non-trivial frequencies into the characteristic frequencies ω_i. The transformation occurs through the rotation frequencies Ω_i of the non-inertial frame, yielding the relation

$$\omega_i = \dot{\overline{\varphi}}_i - \Omega_i, \tag{6}$$

which is exactly the time derivative of equation (3) if we identify the angular frequency Ω_i of the accelerated frame with $-\dot{\Theta}_i$.

3.4. Fictitious trajectory

The key conclusion of the previous subsection is that a perfect equivalence between the Hamiltonians $K = E_{AA}$ and H is fundamentally impossible when non-trivial periodic motion is present. However, full equivalence can be artificially recovered by setting $\Omega = 0$. This condition effectively simulates a transformation between inertial reference frames and leads to a fictitious representation of the trajectory as observed from the inertial frame in which H and the total energy E are originally defined. The resulting trajectory is not physically realized but rather constructed by adjusting the coordinates and momenta so that $\Omega = 0$ is satisfied.

Consequently, we conclude that the AA Hamiltonian K describing non-trivial motion is inherently tied to non-inertial reference frames. Forcing the identity $K = H$ for the canonical transformation $(q, p) \rightarrow (J, \phi)$ can only be accepted if one acknowledges an additional non-canonical transformation within the inertial frame implied by the constraint $\Omega = 0$. Yet, the identity $K = H$ (or $E_{AA} = E$) is customarily presented in educational material as a straightforward result and without mention of the subtleties introduced by non-trivial periodic motion. We argue that a complete understanding of this identity must address the emergence of non-physical (or fictitious) trajectories in the inertial frame. Recognizing the role of non-inertial frames and the relevance of such fictitious trajectories is one of the key pedagogical contributions of this work.

The essential role of inertial effects in determining characteristic frequencies of the non-trivial periodic motion and the deliberate non-canonical elimination of these effects from the energy expression will be further illustrated through the following applied example of the two-dimensional isotropic harmonic oscillator undergoing the non-trivial periodic motion as characterized by Lissajous figures [5].

4. Non-inertial interpretation of AA variables of the 2D isotropic harmonic oscillator

The two-dimensional isotropic harmonic oscillator (2D-IHO) provides the simplest prototype of a broad class of mechanical systems exhibiting non-trivial periodic motion in inertial reference frames. Note, however, that the character of the periodic motion is influenced by the choice of the coordinate system. In Cartesian coordinates, the 2D-IHO periodic motion is trivial in both degrees of freedom and can be decomposed into two uncoupled simple oscillators with their own AA variables (see exercise 9.02 in [21] under the isotropic condition $\omega_x = \omega_y = \omega$). When the same system is reformulated in polar coordinates (r, φ), a different picture emerges. Specifically, a non-trivial periodic motion appears in the angular coordinate φ, which cannot serve directly as one of the angle variables in the AA formalism. At the same time, the radial motion exhibits acceleration and deceleration even though the corresponding radial AA energy $2\omega J_r$ remains constant. This seeming contradiction is resolved by recognizing that the transformation to polar coordinates introduces a non-inertial context. We now explore how the change of coordinates gives rise to a non-inertial setting for the AA variables.

4.1. Non-trivial periodic motion in polar coordinates

In polar coordinates, the conventional calculation of the radial action variable J_r and the polar action variable J_φ leads to the expression of the AA energy $E_{AA} = \omega(2J_r + |J_\varphi|)$ that we assume identical to the inertial energy E in agreement with the ordinary approach (see Supplementary Material SM1 or [21] for a summary of the conventional AA calculation). The characteristic frequencies $\omega_r = 2\omega$ and $\omega_\varphi = \pm\omega$ are obtained via partial derivatives of the energy E_{AA} with respect to J_r and J_φ, respectively.

The characteristic frequencies are seldom derived by directly applying the time derivative to the general definition of angle variables. The route is definitely longer if compared to the derivatives of E_{AA}, but the results reveal the non-inertial nature of the AA variables. The full derivation is summarized in the Supplementary Material SM2 and SM3. Here, we focus on the results under the assumption of $J_\varphi > 0$.

The radial angle variable is ϕ_r and can be written as follows

$$\phi_r = \arcsin[(\rho^2 - \epsilon)/(\epsilon^2 - j_\varphi^2)^{1/2}], \tag{7}$$

where we have defined the dimensionless radial coordinate $\rho = r(m\omega/\sigma)^{1/2}$, with σ being a constant with the units of an angular momentum, and we have also introduced the dimensionless energy $\epsilon = E/(\sigma\omega)$ and the dimensionless polar action variable $j_\varphi = J_\varphi/\sigma$. The inversion of equation (7) yields

$$\rho^2 = \epsilon + (\epsilon^2 - j_\varphi^2)^{1/2} \sin(\phi_r), \tag{8}$$

which shows that the radial periodic motion is trivial. Indeed, the angle variable ϕ_r coincides, up to a constant β_ϵ, with the linearly time-dependent angle coordinate $\varphi_r = 2\omega(t + \beta_\epsilon)$ of the radial motion (compare with Eq. (SM1-4) in the Supplementary Material).

Unlike the radial coordinate, the polar angle φ reveals non-trivial behavior. The definition of equation (2) shows that the polar angle variable consists of two distinct contributions,

$$\phi_\varphi = \varphi + \Theta_\varphi, \tag{9}$$

where the angle Θ_φ is an additional phase term that arises due to the dependences of the radial characteristic function $w_\rho = W_r/\sigma$ on the polar action variable j_φ. Specifically, $\Theta_\varphi = \partial w_\rho/\partial j_\varphi$. As shown elsewhere (see section SM3 in the Supplementary Material), Θ_φ can be decomposed into two contributions

$$\Theta_\varphi = \frac{1}{2}\phi_r + \frac{1}{2}\phi_{j_\varphi}, \tag{10}$$

where ϕ_r is the radial angle variable given in equation (7), and ϕ_{j_φ} is a second angle defined as

$$\phi_{j_\varphi} = \arcsin[(j_\varphi^2/\rho^2 - \epsilon)/(\epsilon^2 - j_\varphi^2)^{1/2}]. \tag{11}$$

Importantly, both φ and Θ_φ are nonlinear functions of time. However, a remarkable cancellation occurs: the nonlinearity in Θ_φ precisely offsets the nonlinearity in φ. To see this explicity, consider the time derivative of equation (10): $\dot{\Theta}_\varphi = \omega(1 - j_\varphi/\rho^2)$ (see equation (SM3-5) in the Supplementary Material). The radial dependence comes from $\dot{\phi}_{j_\varphi} = -2\omega j_\varphi/\rho^2$, where ρ^2 is given by equation (8). Fortunately, $\dot{\varphi} = \omega\partial\epsilon/\partial j_\varphi = \omega j_\varphi/\rho^2$ reveals the hallmark of non-trivial periodic motion finely suppressed in the time derivative of equation (9) and, in the end, $\dot{\phi}_\varphi = \omega$ as expected. Note also that we can conclude that $\varphi = -\phi_{j_\varphi}/2$.

To visualize the non-trivial behavior of the polar coordinate, we refer to figure 1, which compares the time evolutions of φ and $\dot{\varphi}$ with those of ϕ_φ and $\dot{\phi}_\varphi$ for three representative values of the ratio $\eta = j_\varphi/\epsilon$ and for arbitrary initial conditions at $t = 0$. These initial choices do not affect the general meaning of the conlusions.

When $\eta = 1$, the 2D-IHO energy has no contribution from the radial motion. Under this circumstance, the polar coordinate φ coincides exactly with the angle variable ϕ_φ (red solid line in figure 1(a)). In contrast, when $\eta = 0$, the motion is purely radial and the polar coordinate φ is stationary (blue solid lines). For intermediate values of η, the polar angle φ becomes a more sophisticated time-dependent function. In figure 1(a), only the case of $\eta = 0.5$ is considered (black solid line) but similar nonlinear dependences are found for other values of η between zero and one.

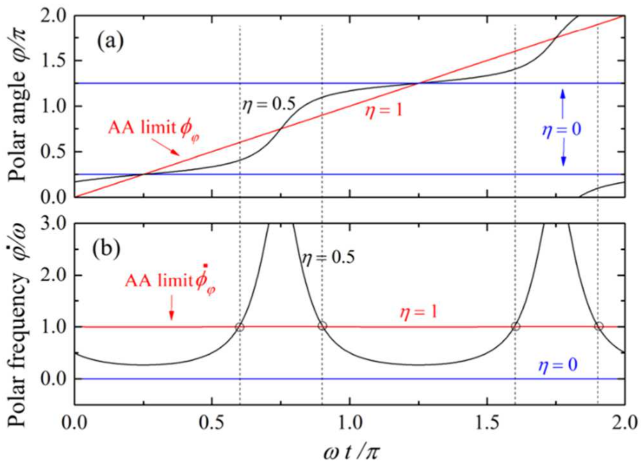

Figure 1. (a) Time dependences of the polar angle φ for three values of the parameter $\eta = j_\varphi/\epsilon$ (η= 0, 0.5 and 1). (b) Time dependences of the frequency $\dot{\varphi}$ for the same values of η. The little black circles in the lower graph define the points where the frequency $\dot{\varphi}$ at $\eta = 0.5$ is equal to the AA value $\dot{\phi}_\varphi = \omega$.

The three distinct regimes of the polar motion are also reflected in the time derivative $\dot{\varphi}$ plotted in figure 1(b). The constant value $\dot{\varphi} = \dot{\phi}_\varphi = \omega$ identifies the trivial circular motion at $\eta = 1$ (red solid line). When $\eta = 0$, the motion is purely radial with zero angular momentum j_φ (blue solid line) and the AA problem reduces to one dimension only. The non-trivial characteristic of the polar motion is manifest for $\eta \neq 1$. Exceptions occur at some isolated points where the constraint $\dot{\varphi} = \dot{\phi}_\varphi$ (or $\dot{\Theta}_\varphi = 0$) is instantaneously satisfied (see the small circles at the intersections of the black and red solid lines). These exceptional points identify the special condition $\Omega = 0$ discussed in the section 3.3. At those points, the angular velocity momentarily matches that of a fictitious circular orbit characterized by $\overline{\rho}^2 = j_\varphi$, where the overbar denotes an idealized trajectory that intersects the real one (i.e. the Lissajous curve) only at those isolated instances. The features generating the trajectory $\overline{\rho}^2 = j_\varphi$ persist for any other choice of η between 0 and 1.

4.2. AA variables as non-inertial observables

The fictitious circular trajectory $\overline{\rho}^2 = j_\varphi$ holds particular significance for the non-inertial interpretation of the oscillator's AA variables. Most notably, it allows us to reinterpret the polar action variable j_φ as the angular momentum of the imaginary uniform circular motion of constant radius $\bar{\rho}$ and constant angular frequency $\dot{\phi}_\varphi = \omega$, as observed from a non-inertial reference frame where the AA variables can be defined.

The non-inertial frame is depicted by the moving blue axes in figure 2, which presents six snapshots extracted from the supplementary video material for the case $\eta = 0.5$ (see the video provided as part of the supporting information). In this figure, the non-inertial frame follows a curvilinear translational motion resulting in an elliptical trajectory (dashed blue line) in the direction opposite to that of the blue arrow. Its tip outlines a uniform motion with constant radius $\overline{\rho}$ and constant angular frequency $\dot{\phi}_\varphi = \omega$. Thus, the curvilinear motion of the non-inertial frame transforms the non-trivial periodic motion observed in the inertial frame (black solid line) into a uniform circular motion within the non-inertial frame. The contrast

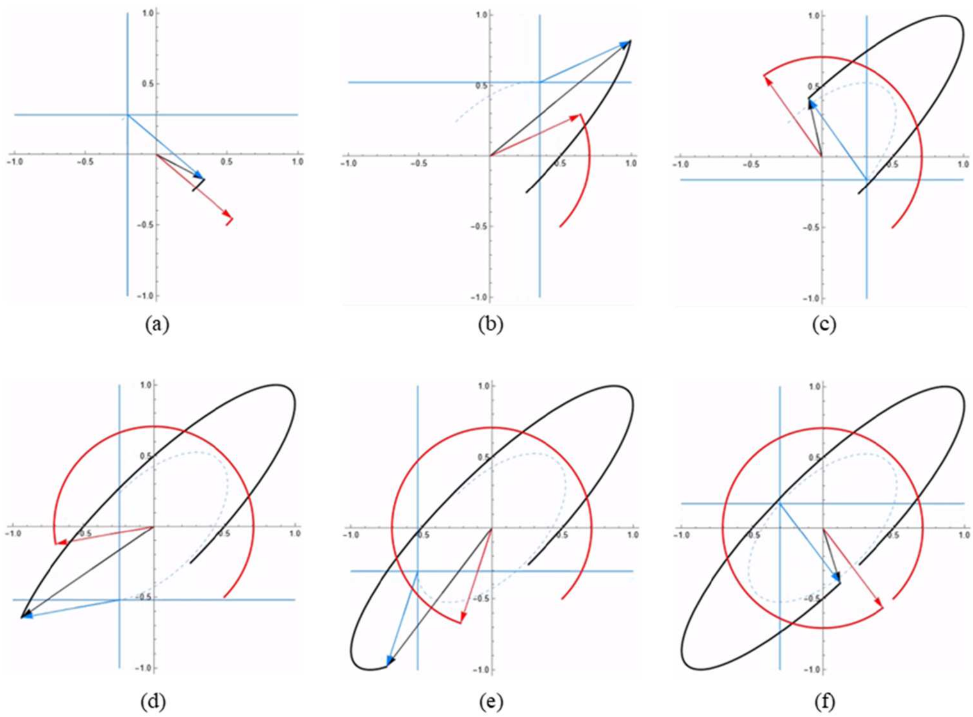

Figure 2. Motion of the 2D isotropic oscillator for $\eta = 0.5$ in two reference frames: inertial (black axes) and non-inertial (blue axes). The tip of the black arrow marks the oscillator's coordinates that generate the Lissajous figure (black solid line) in the inertial frame. The tip of the blue arrow gives the coordinates of the oscillator's orbit in the non-inertial frame whose counter-rotating motion describes an ellipse (dashed blue line). The red arrow traces the apparent AA trajectory known to the observer at rest in the non-inertial frame. Red and blue arrows have the same length and direction at any given time. The complete sequence is extracted from the video provided as supplementary material (see the video provided as part of the supplementary material).

between the inertial (black arrow) and non-inertial (blue arrow) representations of the motion arises from the term $\bar{J} \cdot \Omega$ in the energy transformation of equation (5). Lastly, by selecting points along the real trajectory such that the curvilinear translational motion of the reference frame is removed (i.e. $\Omega = 0$, see previous subsection) one obtains a representation of the non-inertial coordinates $\overline{\rho}$ and ϕ_{φ} in the inertial frame. Although the resulting trajectory (red solid line) is not physically realized, it supports the assumption of energy equivalence between the inertial and non-inertial frames (i.e. $E = E_{\mathrm{AA}}$).

What remains to be clarified, however, is how the AA variables for the radial motion are defined in this context, especially given that the actual trajectory (Lissajous figure) presents accelerations and slowdowns. This question is central to understanding how the full AA framework accommodates both trivial and non-trivial components of the overall dynamics.

Without relying on the calculus-based methods used in the conventional approach, $j_r = J_r/\sigma$ can be derived from the area enclosed by the fictitious radial momentum $\overline{p}_{\rho} = \overline{p}_r/(m\omega\sigma)^{1/2}$ over one period of the trajectory of equation $\overline{\rho}^2 = j_{\varphi}$. The area amounts to $\pi\overline{p}_{\rho}^2/2$. The factor of one-half arises because the radial angle variable ϕ_r in equation (7) traces the circular path twice during one period. Energy conservation in the inertial frame is then $\overline{p}_{\rho}^2 = 2(\epsilon - j_{\varphi})$, which is obtained from the substitution of $\overline{\rho}^2 = j_{\varphi}$ into the HJ equation. Finally, the area becomes $\pi(\epsilon - j_\varphi)$ and we recover the known result for j_r if we divide by 2π. The conclusion is that the virtual circular motion established for $\overline{\rho} = j_\varphi^{1/2}$ is directly related to all of the AA variables.

5. Conclusions

In conclusion, we have shown that while the transformation leading to AA variables poses no issues for trivial periodic motion, the fundamental case of non-trivial periodic motion reveals subtleties that are overlooked in the conventional AA framework, which relies on a passive interpretation of canonical transformations. By adopting an active interpretation, we uncover the necessity of non-inertial reference frames to eliminate the nonlinear time dependence of the angle variables describing motion in the original inertial frame. This cancellation is achieved through the Euler force, encoded in the energy term $\overline{J}{\cdot}\Omega$, which governs the transformation between inertial and non-inertial Hamiltonians. Furthermore, by identifying points along the inertial trajectory where the inertial (or time-dependent) and non-inertial (or constant) frequencies coincide (i.e. $\Omega = 0$), we can extend the representation of uniform circular motion in the inertial frame. Although this reconstructed trajectory is not physically realized, it shares the same energy as the conserved energy in the original inertial frame, thereby justifying the transformation.

The example of the two-dimensional isotropic harmonic oscillator was chosen for two main reasons. One is its simplicity and the other its rich illustrative power in highlighting the role of non-inertial interpretations in the AA formulation of non-trivial periodic motion. Similar insights can be extended to more complex physical systems that exhibit signatures of non-trivial periodic motion. To name just a couple of them, the Kepler–Coulomb problem and the champagne-bottle potential.

From the perspective of this work, the close connection between AA variables and the quantum-mechanical representation of mechanical motion suggests a natural extension of the current non-inertial interpretation to semi-classical and quantum contexts, where the role of inertial effects has received relatively little attention.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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