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Dirac Factorization, Partial/Ordinary Differential Equations and Fractional Calculus

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Abstract

The Dirac factorization method (DFM) is the key feature of the present investigation. It is addressed to the relevant use in diverse fields of research, regarding, e.g., the handling of pseudo-operators arising in quantum mechanics and fractional calculus. We explore the role that the factorization plays in a classical context too, including the study of d'Alembert and Laplace equations. Its strong entanglement with the Cauchy–Riemann conditions and with complex analysis is discussed. We complete our study with the extension of DFM to second-order ordinary differential equations, to classical analytical mechanics, and to higher-order partial differential equations.

Keywords: Dirac factorization; Laplace equation; fractional differential equation; ordinary and partial differential equation; Cauchy–Riemann conditions; pseudo-differential operators

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1. Introduction

The Dirac factorization method (DFM) represents a pivotal development in mathematical physics, laying the groundwork for relativistic quantum mechanics, the rigorous derivation of electron spin, and the theoretical prediction of antimatter [1–3]. DFM has proven particularly effective in addressing pseudo-differential operators that arise in various problems in relativistic quantum mechanics and field theory [4–6].

More recently, its applicability within the framework of fractional calculus has gained attention. Additionally, DFM has emerged as a significant tool in advancing the theory of operator factorization [7–9]. The associated mathematical techniques have been employed to analyze fractional differential equations (FDEs) [10–12], leading to the formulation of the relevant methodology from a different point of view [13–15]. This work explores several aspects arising from this novel perspective, including potential generalizations. It specifically investigates applications of DFM to FDEs and revisits classical equations in mathematical physics, which acquire broader interpretative significance within this framework. For instance, the classical d'Alembert equation is reformulated as a system of first-order partial differential equations (PDEs) [16] with clear physical interpretation in a suitable phenomenological setting. Similarly, the Laplace equation is examined within the same formalism, wherein the theory of multipole expansion [17] emerges naturally.

These results provide a valuable tool for the rigorous evaluation of aberrations in transport of light beams through lens systems and charged beam optics [18–20], namely electron transport through accelerating structures or the guiding of plasma beams. The

Dirac method is further extended to ordinary differential equations, thus yielding new insights into the treatment of phase-space evolution emerging in applications of Hamiltonian dynamics to the previously mentioned topics in optics. The formalism is finally generalized to higher-order PDEs through an extension of the original procedure. By uncovering underlying symmetry structures and offering a new interpretative framework for known physical models, the results contribute to the broader study of operator theory and symmetry in differential equations within mathematical physics [21]. We also explore further important issues emerging from the DFM of the Laplace equation, like, for example, the intimate link with the Cauchy–Riemann conditions and hence with the complex analysis [22–24]. The analysis developed here is endowed with the use of an umbral algebraic formalism, developed during the last years and providing a significant simplification of the analytical [25,26] and numerical computations occurring in this study.

Below, we introduce the DFM [1] using a simple example, limited to the bi-dimensional case. The idea behind its conception can be exemplified, in naïve terms. The procedure, applied to the Pythagorean identity

$$\sqrt{a^2+b^2}=c, \tag{1}$$

allows the elimination of the square root by the use of the linear combination of the a and b, reported below (the superimposed accent denotes a non-ordinary algebraic quantity)

$$\hat{c}=a\,\hat{\sigma}_j+b\hat{\sigma}_k, \quad (j\neq k)\equiv 1,2,3. \tag{2}$$

The $\hat{\sigma}$ operators, appearing in Equation (2), are realized in terms of the Pauli matrices [2], namely

$$\hat{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \hat{\sigma}_2=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\ \hat{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{3}$$

which satisfy the following commutation/anticommutation properties

$$\begin{aligned}
&\hat{\sigma}_j^2=\hat{1},\\
&\left[\hat{\sigma}_j,\hat{\sigma}_k\right]=2i\varepsilon_{j,k,l}\hat{\sigma}_l,\\
&\tfrac{1}{2}\{\hat{\sigma}_j\,,\,\hat{\sigma}_k\}=\delta_{j,k},\\
&[r,s]=rs-sr,\\
&\{r\,,\,s\}=rs+sr,\\
&\hat{1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},
\end{aligned} \tag{4}$$

where ε and δ are the Levi–Civita and Kronecker symbols, respectively, defined as reported below

$$\varepsilon_{j,k,l}=\begin{cases}1 & if\ (j,k,l)\text{ is }(1,2,3),\text{or }(2,3,1),\text{or }(3,1,2),\\ -1 & if\ (j,k,l)\text{ is }(3,2,1),\text{or }(1,3,2),\text{or }(2,1,3)\\ 0 & if\ i=j,\text{or }j=k,\text{or }k=i,\end{cases}, \tag{5}$$

and

$$\delta_{j,k} = \begin{cases} 1 & if\ j = k, \\ 0 & if\ j \neq= k. \end{cases} \tag{6}$$

The identity in Equation (2) is, as shown below, a straightforward consequence of the algebraic rules summarized in (4) and we obtain, indeed,

$$\hat{c}^2 = \left(a\hat{\sigma}_j + b\hat{\sigma}_k\right)\left(a\hat{\sigma}_j + b\hat{\sigma}_k\right) = a^2\hat{\sigma}_j^2 + b^2\hat{\sigma}_k^2 + ab\{\hat{\sigma}_j, \hat{\sigma}_k\} = \left(a^2 + b^2\right)\hat{1}. \tag{7}$$

Accordingly, the price we paid to get rid of the square root is that of replacing "numbers" with matrices.

It is important to underscore that the result is not univocal and depends on the specific values of the indices j and k, chosen to realize the factorization. We have listed a few choices below:

$$\hat{c} = \begin{cases} \begin{pmatrix} 0 & a+ib \\ a-ib & 0 \end{pmatrix} & j = 1, k = 2, \\ \\ \begin{pmatrix} b & a \\ a & -b \end{pmatrix} & j = 1, k = 3, \\ \\ \begin{pmatrix} b & ia \\ -ia & -b \end{pmatrix} & j = 2, k = 3. \end{cases} \tag{8}$$

The result displayed in Equation (8) is reminiscent of the geometrical interpretation of the roots of the unity, once referred to the Gauss–Argand Plane [27].

The geometrical interpretation of the previous identities is summarized in Figure 1. Projecting indeed $\hat{c}$ on a plane (complex or not), we find the following:

1. The decomposition (j = 1, k = 2) can be interpreted as two complex conjugated vectors (Figure 1a)
2. Regarding (j = 1, k = 3), the geometrical image, on the ordinary Cartesian plane, is a couple of vectors with coordinates (b, a), $(a, -b)$ (Figure 1b)
3. Finally, as to the decomposition (j = 2, k = 3), it yields an analogous vector representation on the Argand Plane ($(b, i\,a)$, $(-ia, -b)$) (Figure 1c)

The proper mathematical framing of the previous considerations derives from the properties of the $SU(2)$ group, and we will not dwell on it anymore [21].

The example that follows yields a straightforward account of how DFM can be exploited to transform a pseudo-differential operator [4,6,28] into a fractional derivative. The reason we mentioned the pseudo-differential operator is due to its importance in the original DFM, which led to the formulation of the Dirac equation [1–3].

Before going further, we should specify what the relevance of the DFM is to the technicalities of the fractional calculus [10–12]. To this aim, we consider the pseudo-differential operator

$$\hat{O} = \sqrt{1 + \partial_x}, \tag{9}$$

which, according to the previous prescriptions, can be written as

$$\hat{O} = \hat{\sigma}_j + \hat{\sigma}_k \partial_x^{\frac{1}{2}}, \tag{10}$$

It is therefore evident that

$$\hat{O}_{1,2} = \begin{pmatrix} 0 & 1 + i\partial_x^{\frac{1}{2}} \\ 1 - i\partial_x^{\frac{1}{2}} & 0 \end{pmatrix}, \tag{11}$$

which, acting on the unit vector, yields

$$\hat{O}_{1,2}\hat{1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 + i\partial_x^{\frac{1}{2}} \\ 1 - i\partial_x^{\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \frac{i}{\sqrt{\pi x}} \\ 1 - \frac{i}{\sqrt{\pi x}} \end{pmatrix}, \tag{12}$$

(recall that according to the Euler definition $\partial_x^{\frac{1}{2}} 1 = \frac{1}{\sqrt{\pi x}}$; see ref. [10])

In the following, we draw further consequences from the previously outlined formalism, which sheds light on the theory of factorization of partial differential equations [7–9,29–36], opening new perspectives that are worth pursuing.

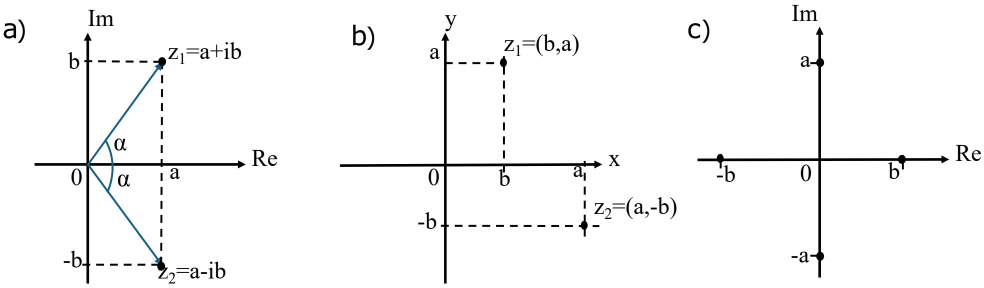

Figure 1. Argand Plane geometrical representation of the operator $\hat{O}_{1,2}$ on the unit vector.

Before proceeding further, it is worth framing the present research within the context of previous studies. Apart from the references [1,2,7–10,20,27] representing the backbone of our forthcoming discussion, we would like to emphasize that factorization methods have played a prominent role in classical and quantum mechanics. At the end of the 19th century, geometrical speculations on the properties of the second-order differential equations anticipated aspects of the quantum formalism [6], which were later considered within a more specific physical context in [31–36]. The use of specific factorization invariants introduced in [28] has been reformulated in refs. [31–36]. In these contributions, the theory of Darboux invariants has been reformulated in terms of the Dirac spinor formalism, and a bridge has been established between second-order ODEs with non-constant coefficients, their geometrical content, and a set of bi-unitary transformations. The point of view developed here is a contribution towards an effort of putting within the same mathematical context a number of apparently uncorrelated topics (Darboux theory, diffusion equations, Laplace equation ...). The keynote of this analysis is the use of DFM endowed with advanced operational techniques.

2. Dirac Factorization, Fractional Calculus, and Equations of Mathematical Physics

In Figure 2, we report the ***"relativistic triangle"***, representing a geometrical representation of the special relativity kinematic identity [18,19]

$$E^2 = \left(m_0 c^2\right)^2 + (pc)^2, \tag{13}$$

which links the energy of a particle to its momentum and rest mass. Although this geometrical picture expresses a naïve coincidence between mathematical entities, it can sometimes be useful to establish further identities. It is indeed worth noting that

$$\begin{aligned} cos(\theta) &= \frac{m_0c^2}{E} = \frac{1}{\gamma}, \\ cos(\varphi) &= sin(\theta) = \frac{pc}{E} = \beta. \end{aligned} \tag{14}$$

The properties of this triangle are widely discussed in the specialized literature and will not be discussed here any further.

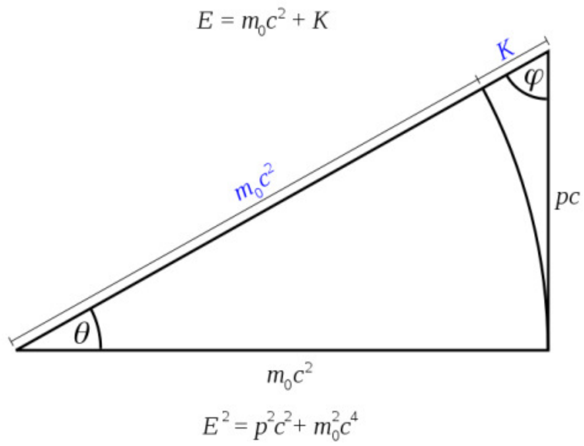

Figure 2. Relativistic triangle.

Considering that the identity in Equation (13) is Pythagorean and applying the DMF, we find

$$\hat{E} = \left(m_0c^2\right)\hat{\sigma}_3 + (pc)\hat{\sigma}_1 = \begin{pmatrix} m_0c^2 & pc \\ pc & -m_0c^2 \end{pmatrix} = m_0\gamma c^2 \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}, \tag{15}$$

and

$$\hat{E} = \left(m_0c^2\right)\hat{\sigma}_1 + (pc)\hat{\sigma}_2 = \begin{pmatrix} 0 & m_0c^2 + ipc \\ m_0c^2 - ipc & 0 \end{pmatrix} = m_0\gamma c^2 \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}. \tag{16}$$

Identities written in a suggestive geometrical fashion, but straightforward consequence of the identities in Equation (13). Avoiding any further speculation in this respect, we interpret energy and momentum in terms of quantum-mechanical operators, we eventually end up with the wave equation

$$i\hbar\partial_t\underline{\psi} = \left(m_0c^2\right)\hat{\sigma}_3\underline{\psi} - i\hbar c\partial_x\hat{\sigma}_1\underline{\psi}, \tag{17}$$

where $\underline{\psi}$ is a two-component column vector, and Equation (17) is the 1D Dirac equation. The different realizations of the identity (17) change the form of the equation but not its physical meaning.

The solution of Equation (17) can indeed be written as

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = e^{-i\tau\begin{pmatrix} 1 & -i\partial_\xi \\ -i\partial_\xi & -1 \end{pmatrix}} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}_{\tau=0}, \quad \tau = \frac{m_0c^2}{\hbar}t, \; \xi = \frac{m_0c}{\hbar}x \tag{18}$$

where $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}_{t=0}$ represents the "initial condition" of the Cauchy problem in Equation (17) and is a function of the spatial coordinate only.

The evolution operator on the right-hand side of Equation (18) realizes a rotation matrix, determining a kind of oscillation between up and down components $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$. Further comments can be found in ref. [6], where the Authors have addressed the problem of the evolution of two-component Pauli-Dirac equations. In Appendix A we review the solution formalism and comment on the application of the method to the so-called relativistic heat equation.

We have reviewed the essential elements characterizing the DFM. In the following, we use and generalize the procedure to more complex cases. We like to emphasize the importance of the Dirac proposal. It has benefited from abstract speculation due to Clifford and other mathematicians of the 19th century, and it has turned into a powerful tool for exploring physical reality.

3. Generalities on Factorization Methods

The essential elements of the DFM exploited for the purposes of this article are fixed in refs. [13–16] and are summarized below for readers' convenience. Given the equation

$$\hat{L}u = 0, \tag{19}$$

where the differential operator $\hat{L}$ is expressible as

$$\hat{L} = \hat{P} \cdot \hat{Q}, \tag{20}$$

with $\hat{P}$, $\hat{Q}$ defining the further differential equations

$$\hat{P}u_1 = 0, \; \hat{Q}u_2 = 0. \tag{21}$$

Accordingly, the solutions $u_{1,2}$ and u are linked by

$$u_1 + u_2 = u, \tag{22}$$

if $\hat{P}$ and $\hat{Q}$ are commuting operators.

Without discussing the conditions to be satisfied by the operators $\hat{L}, \; \hat{P}, \; \hat{Q}$ to realize the factorization (20), we show how the DFM can be framed within an analogous context. A useful example is provided by ref. [16]. We consider the d'Alembert equation

$$\partial_x^2 \psi - \frac{1}{c^2} \partial_t^2 \psi = 0, \tag{23}$$

and applying the previously outlined procedure

$$\begin{aligned} &\left(\hat{\sigma}_j \partial_x + i \hat{\sigma}_k \partial_\tau\right) \left(\hat{\sigma}_j \partial_x + i \hat{\sigma}_k \partial_\tau\right) \psi = 0, \\ &\tau = ct. \end{aligned} \tag{24}$$

We find

$$\begin{aligned} &(\hat{\sigma}_j \partial_x + i \hat{\sigma}_k \partial_t) \underline{\phi} = 0, \\ &\underline{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \neq \psi. \end{aligned} \tag{25}$$

Considering the case $j = 1,\ k = 2$, we find

$$\begin{pmatrix} 0 & \partial_x + \partial_\tau \\ \partial_x - \partial_\tau & 0 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} = 0,$$
$$\phi_\pm = f(x \mp t). \tag{26}$$

The conclusion is therefore that the solution of the original problem in Equation (23) is recognized to be provided by the components $\phi_\pm$, understood as the d'Alembert forward and backward solutions

$$\phi_\pm = f(x \mp t) \tag{27}$$

The DFM applied to the study of other classical PDE of mathematical physics gives rise to interesting surprises since the resulting components might result to being ruled by fractional derivative equations (FDE).

An example is provided by the heat equation [37]

$$\partial_t u = K \partial_x^2 u, \tag{28}$$

which treated with the paradigm we have outlined, opens the possibility of developing further interesting speculations, going beyond the case we discussed in the case concerning the d'Alembert equation. Indeed, we find

$$\left(\hat{\sigma}_j \partial_t^{1/2} + i\sqrt{K} \hat{\sigma}_k \partial_x \right) \underline{v} = 0, \tag{29}$$

and specifying it for the indices $j = 1,\ k = 2$ yields the component equations

$$\begin{aligned} \partial_t^{1/2} v_1 - \sqrt{K} \partial_x v_1 = 0, \\ \partial_t^{1/2} v_2 + \sqrt{K} \partial_x v_2 = 0. \end{aligned} \tag{30}$$

Albeit achievable with more conventional means, this couple of solutions v_1, v_2 to the commented ones can be viewed as the constitutive "elements" of the heat equation, which originates from the second Fick's law [38–40], which establishes a linear relationship between the molar concentration (c) and the diffusion flux (J), namely

$$\partial_t c + \partial_x J = 0. \tag{31}$$

The noticeable conclusion is that the factorization of the heat equation has led to a pair of fractional equations with definite physical meaning.

Considering the first of Equation (30), we can attempt the following interpretation $J = -v_1 \sqrt{K}$, furthermore, the molar concentration can be linked to v_1 through the identity $c = \partial_t^{-\frac{1}{2}} v_1$. This last relationship opens some speculations on the role of the memory kernel in diffusion processes. The molar concentration can indeed be written in terms of the component v_1 in terms of the Riemann–Liouville fractional integral [11,12,41]

$$c(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t \frac{v_1(x, \tau)}{\sqrt{t - \tau}} d\tau, \tag{32}$$

which shows that the evolution of c depends on the memory kernel, associated with the time history of v_1.

A similar interpretation emerges in the treatment of the two-component heat equation, discussed in Appendix A.

Regarding the Schrödinger equation, a Dirac factorized form can be written as

$$\left(\hat{\sigma}_j\partial_{\bar{t}}^{1/2}+\sqrt{\frac{i}{2}}\hat{\sigma}_k\partial_{\bar{x}}\right)\underline{\psi}=0,\ \ \bar{t}=\frac{2\pi ct}{\lambda_c},\ \ \bar{x}=\frac{2\pi x}{\lambda_c},\ \ \lambda_c=\frac{h}{mc},\tag{33}$$

choosing $j = 1$ and $k = 3$, the previous identity reduces to the following set of coupled equations

$$\begin{pmatrix}\sqrt{\frac{i}{2}}\partial_{\bar{x}} & \partial_{\bar{t}}^{1/2}\\ \partial_{\bar{t}}^{1/2} & -\sqrt{\frac{i}{2}}\partial_{\bar{x}}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0.\tag{34}$$

The above equation can be further handled and eventually yields

$$\sqrt{\frac{i}{2}}\partial_{\bar{x}}\psi_1=\partial_{\bar{t}}\varphi,\ \ \varphi=-\frac{1}{\sqrt{\pi}}\int_0^t\frac{\psi_2(\tau)}{\sqrt{t-\tau}}d\tau.\tag{35}$$

The presence of fractional derivatives suggests that the components of the Schrödinger equations may be associated with anomalous diffusion mechanisms [42].

The point we like to underscore is the strong entanglement between DFM and coupled equations containing half order derivatives. Regarding the Schrödinger equation, this aspect of the problem was discussed by Levy-Leblond in the 1960s of the last century [43]. He pointed out that such a correspondence raises the question that spin and electron anomalous magnetic moment are properties not of a genuine relativistic nature. The significant novelty of the results of this type is just the possibility of establishing a link with an equation of fractional nature, having not only a mere mathematical interest but a definite physical meaning. We will not dwell on the relevant solution, which can be obtained using methods discussed in specialized textbooks [10–12,44].

We have so far limited the discussion to derivatives and the number of members of the equations not larger than two. In the forthcoming sections, we extend the validity of our analysis. We show that the applicability of the method to PDE, including derivatives of order larger than 2, requires the use of a higher-order realization of the Pauli matrices.

4. Multidimensional Equations and Dirac Factorization

In the previous sections, we showed that DMF leads to a non-conventional, non-trivial factorization method for PDEs, often encountered in mathematical physics. If we stretch the procedure further, new and interesting results emerge.

Let us therefore consider the following d'Alembert equation with a "damping" term (provided by a first order derivative in time; see ref. [16])

$$\partial_x^2\psi-\frac{1}{c^2}\partial_t^2\psi+\kappa\partial_t\psi=0.\tag{36}$$

The above equation can be regarded as a heuristic modification of a nondissipative wave equation in which damping has been included phenomenologically (for further comments, see ref. [45]).

The Dirac factorized form of Equation (36) reads

$$\left(\hat{\sigma}_j\partial_x+i\hat{\sigma}_k\partial_\tau+\sqrt{\kappa c}\hat{\sigma}_l\partial_\tau^{1/2}\right)\left(\hat{\sigma}_j\partial_x+i\hat{\sigma}_k\partial_\tau+\sqrt{\kappa c}\hat{\sigma}_l\partial_\tau^{1/2}\right)\psi=0,\tag{37}$$

with $\tau = ct$. Choosing $(j,k,l) = (1,2,3)$

$$\begin{pmatrix}\sqrt{\kappa c_l}\partial_\tau^{1/2} & \partial_x+\partial_\tau\\ \partial_x-\partial_\tau & -\sqrt{\kappa c_l}\partial_\tau^{1/2}\end{pmatrix}\begin{pmatrix}\varphi_+\\ \varphi_-\end{pmatrix}=0,\tag{38}$$

we eventually find

$$\begin{aligned}(\partial_x+\partial_\tau)\,\varphi_- &= -\sqrt{\kappa\, c}\partial_\tau^{1/2}\varphi_+,\\ (\partial_x-\partial_\tau)\,\varphi_+ &= \sqrt{\kappa\, c}\partial_\tau^{1/2}\varphi_-.\end{aligned} \tag{39}$$

In this case, too, integer- and fractional-order phenomenology are entangled, and conventional methods may hardly reveal this mathematical property.

Going back to the original meaning of the equation, describing electromagnetic waves propagating in absorptive media, Equation (34) may symbolize the fact that, during the absorption process, the progressive wave is affected by the regressive wave (and vice versa).

The presence of derivatives of order ½ can, however, be avoided if other DFM strategies are adopted, as discussed in Appendix A.

The methods we have just outlined can be extended to other classical equations of mathematical physics. The two-dimensional Laplace equation writes

$$\left(\partial_x^2+\partial_y^2\right)\psi(x,y)=0. \tag{40}$$

The relevant solutions can be written in terms of the so-called normal (ψ_n^N) and skew $\left(\psi_n^S\right)$ multipoles [17,18], which can be compactly written as

$$\psi_n^N=n!\sum_{r=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^r x^{n-2r}y^{2r}}{(n-2r)!(2r)!}, \tag{41}$$

and

$$\psi_n^S=-i\,n!\sum_{r=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\frac{(-1)^r x^{n-(2r+1)}y^{2r+1}}{(n-2r)!(2r+1)!}, \tag{42}$$

which look like the two variable Hermite polynomials [25] and expressible as ordinary Newton binomials, namely

$$\begin{aligned}\psi_n^N &= \left(x+i\hat{h}_N y\right)^n f_0,\\ \psi_n^S &= -i\left(x+i\hat{h}_S y\right)^n f_0.\end{aligned} \tag{43}$$

where $\hat{h}_{N,S}$ are umbral operators which, acting on the vacuum f_0 [25,26], yield

$$\left(\hat{h}_N\right)^r f_0=\left|cos\left(\frac{\pi r}{2}\right)\right|,\ \left(\hat{h}_S\right)^r f_0=\left|sin\left(\frac{\pi r}{2}\right)\right|. \tag{44}$$

Thus, eventually, yielding the following alternative form of Equation (43)

$$\begin{aligned}\psi_n^N &= \sum_{r=0}^{n}\binom{n}{r}x^{n-r}i^r y^r cos\left(\frac{\pi r}{2}\right),\\ \psi_n^S &= -i\sum_{r=0}^{n}\binom{n}{r}x^{n-r}i^r y^r sin\left(\frac{\pi r}{2}\right).\end{aligned} \tag{45}$$

The Laplace equation in Dirac factorized form, involving, e.g., $\hat{\sigma}_{1,3}$ operators, assumes the matrix form

$$\left(\hat{\sigma}_1\partial_x+\hat{\sigma}_3\partial_y\right)\begin{pmatrix}\psi\\ \psi^*\end{pmatrix}=\begin{pmatrix}0 & \partial_x+i\partial_y\\ \partial_x-i\partial_y & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi^*\end{pmatrix}=0, \tag{46}$$

equivalent to the Cauchy–Riemann conditions [46], concerning the real and imaginary parts of the ψ function, recognized as

$$Re(\psi)=\psi_n^N,\ \ Im(\psi)=\psi_n^S. \tag{47}$$

The link between monogeneity conditions and the Dirac method is not new, albeit not as widely known as it should be. Its profound implications have been discussed in the past (see refs. [22–24]), and we comment further on it in the final section.

The search for the solution of the three-dimensional Laplace equation

$$\vec{\nabla}^2\psi(x,y,z) = \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\psi(x,y,z) = 0, \tag{48}$$

can be afforded by an extension of the method of multipoles. The three-dimensional generalization of the functions $\psi_n^{N,S}$ requires the inclusion of the additional variable z, and the use of the formalism, we have foreseen, allows the definition of the N three-dimensional multipoles, in the form of the "Newton trinomial"

$$\psi_n^N = \left(x + {}_1\hat{h}_N y + {}_2\hat{h}_N z\right)^n {}_1\varphi_0 {}_2\varphi_0, \tag{49}$$

where the operators ${}_{1,2}\hat{h}_N$ act separately on the vacua ${}_{1,2}\varphi_0$. In non-umbral terms, we find that

$$\psi_n^N(x,y,z) = \sum_{r=0}^{n}\binom{n}{r}\psi_{n-r}^N(x,-iy)(i\sqrt{2}z)^r\left|\cos\left(\frac{\pi}{2}r\right)\right|, \tag{50}$$

and easily prove that they satisfy the following identity

$$\left(\partial_x^2 + \partial_y^2\right)\psi_n^N(x,y,z) = -\partial_z^2\psi_n^N(x,y,z), \tag{51}$$

analogous conclusions are obtained for the skew counterparts. The three-dimensional Laplace equation can be factorized using the three Pauli matrices (e.g., $\left(\hat{\sigma}_1\partial_x + \hat{\sigma}_2\partial_y + \hat{\sigma}_3\partial_z\right)\underline{\psi} = 0$), yielding the corresponding matrix form

$$\begin{pmatrix} \partial_z & \partial_x + i\partial_y \\ \partial_x - i\partial_y & -\partial_z \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \tag{52}$$

for which the components $\psi_{1,2}$ have the same interpretation given before (for the computational details, see Appendix A).

Further extension of the method to the three-dimensional Laplace is discussed in the next section, where we comment further on the link between monogeneity conditions and the Dirac method.

In this section, we covered different aspects of the Dirac factorization method applied to the theory of partial differential equations occurring in classical mathematical physics, hence involving second-order derivatives. In the following section, we show that the method can be extended to higher orders.

5. Extension of the DFM to Higher-Order Derivatives

In this section, we extend the DFM to PDE with a derivative order larger than 2. We therefore need an identity extending to third-order power in Equation (2). The price paid to accomplish this task is slightly more than before, since the use of 3 × 3 matrices is in order.

To enter abruptly into the specific topic of this section, we state the identity [14]

$$\hat{c}^3 = \left(a\hat{\tau}_j + b\hat{\tau}_k\right)\left(a\hat{\tau}_j + b\hat{\tau}_k\right)\left(a\hat{\tau}_j + b\hat{\tau}_k\right) = \left(a^3 + b^3\right)\hat{1}, \tag{53}$$

and look for the realization of the τ-operators which, in analogy with the Pauli matrices, allow the fulfillment of the previous identity. The expansion of the products in Equation (53) yields the conditions to be satisfied, namely

$$\hat{\tau}_j^3 = \hat{1}, \quad \hat{\tau}_j^2\hat{\tau}_k + \hat{\tau}_j\hat{\tau}_k\hat{\tau}_j + \hat{\tau}_k\hat{\tau}_j^2 = 0. \tag{54}$$

The cubic roots of unity are therefore involved in the process of determining the τ-operators, expressible in terms of 3×3 matrices. As shown in ref. [14], they belong to an eight-dimensional Clifford algebra, and two of the representative set of eight matrices (see Appendix A) write

$$\hat{\tau}_1 \equiv \tau_j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{\tau}_2 \equiv \tau_k = \begin{pmatrix} 0 & \epsilon_+ & 0 \\ 0 & 0 & \epsilon_- \\ 1 & 0 & 0 \end{pmatrix}, \tag{55}$$

where we denote by $\epsilon_\pm$ the quantities reported below and link to the roots of unity through the identities reported below

$$\begin{aligned} &\epsilon^3 = 1, \;\; \epsilon_0 + \epsilon_+ + \epsilon_- = 0, \;\; \epsilon_+ \epsilon_- = 1, \\ &\epsilon_0 = 1, \;\; \epsilon_\pm = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \end{aligned} \tag{56}$$

The validity of Equation (53) is easily checked by directly plugging the identities (55) in Equation (54) and using (56).

We apply the DFM to pseudo-operators involving cubic roots, a naïve extension of Equations (9) and (10) yields

$$\hat{O} = \sqrt[3]{1+\partial_x} \to \tau_1 + \tau_2 \partial_x^{\frac{1}{3}} = \begin{pmatrix} 0 & 1+\epsilon_+\partial_x^{\frac{1}{3}} & 0 \\ 0 & 0 & 1+\epsilon_-\partial_x^{\frac{1}{3}} \\ 1+\partial_x^{\frac{1}{3}} & 0 & 0 \end{pmatrix} \equiv \hat{O}_{1,2}, \tag{57}$$

and

$$\hat{O}_{1,2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\epsilon_+}{\Gamma\left(\frac{2}{3}\right)} \frac{1}{\sqrt[3]{x^2}} \\ 1 + \frac{\epsilon_-}{\Gamma\left(\frac{2}{3}\right)} \frac{1}{\sqrt[3]{x^2}} \\ 1 + \frac{1}{\Gamma\left(\frac{2}{3}\right)} \frac{1}{\sqrt[3]{x^2}} \end{pmatrix}. \tag{58}$$

The last identity represents a generalization of the Argand–Gauss complex numbers representation.

The use of the "cubic" factorization is applied, in the following, to PDE containing derivatives of order 3. To proceed, assume that the conditions stated in ref. [16] hold for derivatives, including orders larger than 2.

The "third-order Laplace" equation

$$\left(\partial_x^3 + \partial_y^3\right)\psi(x,y) = 0, \tag{59}$$

provides an example clarifying how the DFM works for higher-order derivatives PDEs. The factorization of Equation (59) in terms of 3×3 matrices writes

$$\begin{pmatrix} 0 & \partial_x + \epsilon_+\partial_y & 0 \\ 0 & 0 & \partial_x + \epsilon_-\partial_y \\ \partial_x + \partial_y & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0. \tag{60}$$

The solution in terms of generalized multipoles will be discussed in this section and in Appendices A and B, where we provide the computational details.

The third-order ${}_3\psi_n^N$ multipoles, providing one of the solutions of the Laplace Equation (60), in polynomial form, read

$$ {}_{(3,0)}\psi_n^N(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{3} \rfloor} \frac{(-1)^r x^{n-3r} y^{3r}}{(n-3r)!(3r)!}, \tag{61} $$

and its Newton binomial umbral image can be written as

$$ {}_{(3,0)}\psi_n^N(x,y) = \left(x + y_3\hat{h}\right)^n \varphi_0 \,, \quad {}_3\hat{h}^r \varphi_0 = \delta_{m\lfloor \frac{r}{m} \rfloor, r}, \tag{62} $$

the skew counterparts are, respectively, given by

$$ \begin{aligned} {}_{(3,1)}\psi_n(x,y) &= n! \textstyle\sum_{r=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{(-1)^r x^{n-3r-1} y^{3r+1}}{(n-3r-1)!(3r+1)!}, \\ {}_{(3,2)}\psi_n(x,y) &= n! \textstyle\sum_{r=0}^{\lfloor \frac{n-2}{3} \rfloor} \frac{(-1)^r x^{n-3r-2} y^{3r+2}}{(n-3r-2)!(3r+2)!}, \end{aligned} \tag{63} $$

reminiscent of the higher-order Hermite polynomials discussed in [25,47].

The solution of Equation (60) can be accordingly written as

$$ \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \equiv \begin{pmatrix} {}_{(3,0)}\psi_n^N \\ {}_{(3,1)}\psi_n^N \\ {}_{(3,2)}\psi_n^N \end{pmatrix}. \tag{64} $$

Specific details are given in Appendix A, where we discuss the solution of the third-order three-variable Laplace equation, along with a note on the higher-order cases.

6. Factorization Methods and Ordinary Differential Equations

In this article, we touched on different aspects concerning the use of DFM in a context regarding the solution of PDE, emerging in a quantum and/or classical context. We underscored that these methods open interesting elements of research regarding the entanglement between fractional, "conventional" calculus and the possibility of exploring the consequent physical implications. The results we obtained, rather than giving organic solutions, propose organic elements of discussion in a field of research that deepens its roots in important studies dating back to more than one hundred years.

On the eve of the 19th century, Sophus Lie started an ambitious program [48], aimed at exploiting the methods of group theory to state the solvability of ordinary differential equations. The contributions of Darboux [29] were directed along the same line (albeit not directly characterized by the use of group theoretic methods), subsequently these studies have been pursued with the factorization methods of Dirac [1], Schrödinger [7,8] Infeld–Hull [9], Feshbach–Villars [49] and eventually with the researches concerning the link between Cauchy–Riemann conditions and Dirac operators [22,46,50], which evolved into the development of the theory of hyper-complex (quaternionic) functions and Dirac-Fueter equations [23,24].

Although most of the quoted factorization techniques have been developed to treat ordinary differential equations (ODEs) with non-constant coefficients, our discussion has been limited to PDEs. We underscore that, within this context, a sharp cut cannot be drawn, and the studies in refs. [7–9] have been addressed to second-order ODEs, originated by the solution of Schrödinger equations with specific potentials (see [9]). To fill the gap, we devote a part of this final section to commenting on how the methods concerning the factorization of ODE with non-constant coefficients are naturally embedded in those discussed in the first part of the article.

The starting point of the forthcoming discussion is the observation that any second-order differential equation

$$y^{''} + a(x)y' + b(x)y = 0, \;\; (') = \frac{d}{dx}, \quad ('') = \frac{d^2}{dx^2}, \tag{65}$$

can be factored into a two-component equation

$$\frac{d}{dx}\begin{pmatrix} y_+ \\ y_- \end{pmatrix} = \begin{pmatrix} -a(x) & -b(x) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_+ \\ y_- \end{pmatrix}, \;\; y_+ = y', \;\; y_- = y, \tag{66}$$

which can be further handled, using the transformation (the lower limit α is an arbitrary constant and does not affect the conclusions drawn below)

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = e^{-\frac{1}{2}\int_\alpha^x a(x')dx'}\begin{pmatrix} y_+ \\ y_- \end{pmatrix}, \tag{67}$$

which reduces it to the traceless form

$$\frac{d}{dx}\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \hat{A}(x)\begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \tag{68}$$

being the matrix on the right-hand side of Equation (68) a linear combination of the three Pauli matrices (which are also the generators of the $SU(2)$ algebra), namely

$$\hat{A}(x) = \begin{pmatrix} -\frac{a}{2} & -b \\ 1 & \frac{a}{2} \end{pmatrix} = -\frac{a(x)}{2}\hat{\sigma}_3 + \frac{1-b(x)}{2}\hat{\sigma}_1 + i\frac{1+b(x)}{2}\hat{\sigma}_2, \tag{69}$$

the solution of Equation (68) can be obtained using the evolution operator method and ordering techniques, typical of quantum mechanics (see refs. [20,31–33,51,52] for the specific details). Within this context, it has been shown that the explicit solution depends on that of the following Riccati equation

$$\begin{aligned} &k' - k^2 + Q_+(x) = 0, \\ &Q_+(x) = \frac{a^2}{4} + \frac{a'}{2} - b, \end{aligned} \tag{70}$$

u_- in Equation (69) satisfying the second-order equation

$$u_-^{''} - Q_+(x)u_- = 0, \tag{71}$$

usually referred to as the Liouville standard form.

Along with Equation (68) the following equation is introduced

$$\frac{d}{dx}\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = -\hat{A}^T(x)\begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \tag{72}$$

where $\hat{A}^T(x)$ is the transpose of the $\hat{A}(x)$ matrix. The importance of the two-component vector v is manifold.

The scalar product $v^T u$ is an invariant, whose role in physical problems has been discussed in [31–33]. Furthermore (if b is independent of x) it can be shown that v_-satisfies the Liouville standard form

$$v''_- - Q_-(x)v_- = 0,\ Q_-(x) = \frac{a^2}{4} - \frac{a'}{2} - b. \tag{73}$$

If interpreted in terms of Schrödinger equations, the Equations (71) and (73) are characterized by the two "super potentials" $Q_{\mp}(x)$ mentioned in supersymmetric quantum mechanics [15].

The last discussion is apparently different from that of the first part of the article, where we have applied the DFM paradigm to reduce the order of a PDE. To get a more direct comparison, we write Equation (65) as reported below

$$\begin{gathered}\left(\frac{d}{dx} + \tfrac{1}{2}a(x)\right)^2 y - \hat{S}(x)y = 0,\\ \hat{S}(x) = \left(Q_+ + \frac{1}{2}a(x)\frac{d}{dx}\right),\end{gathered} \tag{74}$$

applying the DFM, as illustrated in the introductory section, we extract the square root of the right-hand side of Equation (74) and find the identity

$$\left[\left(\frac{d}{dx} + \frac{1}{2}a\right)\hat{\sigma}_1 + i\sqrt{Q_0}\hat{\sigma}_2\right]\varphi\ = 0,\ \ Q_0 = \frac{a^2 - 4b}{4}, \tag{75}$$

which holds if a and b are not depending on the variable x. We therefore find

$$\begin{pmatrix} 0 & \left(\frac{d}{dx} + \frac{1}{2}a\right) - \sqrt{Q_0} \\ \left(\frac{d}{dx} + \frac{1}{2}a\right) + \sqrt{Q_0} & 0 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 0, \tag{76}$$

where the components $\varphi_{1,2}$ are the independent solution of a second-order differential equation with constant coefficients. An analogous factorization is not admitted for non-constant coefficients, since $\left[\left(\frac{d}{dx} + \frac{1}{2}a\right), Q_+(x)\right] \neq 0$.

We overcome the problem of the non-commutativity by considering the following auxiliary operator:

$$\begin{gathered}\hat{\Sigma} = \frac{\Omega(x)\widehat{\sigma}_1 - \widehat{\sigma}_2}{\sqrt{2}i}\frac{d}{dx} = \hat{\Sigma}_- + \hat{\Sigma}_+,\\ \hat{\Sigma}_- = \begin{pmatrix} 0 & \hat{A}^- \\ 0 & 0 \end{pmatrix},\ \hat{\Sigma}_+ = \begin{pmatrix} 0 & 0 \\ \hat{A}^+ & 0 \end{pmatrix},\\ \hat{A}^+ = \tfrac{1}{\sqrt{2}}\left(\Omega(x) - \tfrac{d}{dx}\right),\ \ \hat{A}^- = \tfrac{1}{\sqrt{2}}\left(\Omega(x) + \tfrac{d}{dx}\right),\\ \left[\hat{A}^+, \hat{A}^-\right] = \hat{A}^+\hat{A}^- - \hat{A}^-\hat{A}^+ = \Omega'(x),\end{gathered} \tag{77}$$

where the operators $\hat{A}^{\pm}$ reduce to the creation and annihilation operators of the quantum harmonic oscillator for $\Omega(x) = x$.

Squaring the operator $\hat{\Sigma}$, we find

$$\begin{aligned}\hat{\Sigma}^2 &= \hat{H} + \frac{i}{2}\Omega'(x)\hat{\sigma}_1\hat{\sigma}_2, \\ \hat{H} &= \left[-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\Omega^2(x)\right]\hat{1},\end{aligned} \tag{78}$$

where the operator $\hat{H}$ can be viewed as the Hamiltonian of a position-dependent harmonic oscillator written as

$$\hat{H} = \hat{\Sigma}^2 - \frac{i}{2}\Omega'(x)\hat{\sigma}_1\hat{\sigma}_2 = \begin{pmatrix} \hat{A}^-\hat{A}^+ - \frac{\Omega'(x)}{2} & 0 \\ 0 & \hat{A}^+\hat{A}^- + \frac{\Omega'(x)}{2} \end{pmatrix}. \tag{79}$$

The matrices $\hat{\Sigma}_\mp$ are known as Super-Charges associated with the Super-Potentials $\Omega_\pm = \Omega \pm \frac{1}{2}\Omega'$ [20,51,52].

The conclusion we may draw from this discussion is that DFM is a mathematical technique that allows the unification of different concepts ranging from a purely mathematical context to classical and quantum mechanics.

Before closing this section, we consider the following two-dimensional partial differential equation

$$\begin{aligned} i\partial_z\psi &= \hat{T}\psi, \\ \hat{T} &= -\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) + \frac{n(x,y)}{2},\end{aligned} \tag{80}$$

which rules the paraxial propagation of a light beam moving in a medium with a position-dependent refractive index [53]. From the mathematical point of view, Equation (80) is a Schödinger equation, which can be treated by using the factorization procedure leading to Equation (79).

Introducing the operator

$$\begin{aligned}\hat{\Sigma} &= \frac{\sqrt{n}\hat{\sigma}_1 - i\hat{\sigma}_2\partial_x - i\hat{\sigma}_3\partial_y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i\partial_y & \hat{A}^- \\ \hat{A}^+ & i\partial_y \end{pmatrix}, \\ \hat{A}^- &= \sqrt{n} + \partial_x, \quad \hat{A}^+ = \sqrt{n} - \partial_x,\end{aligned} \tag{81}$$

which, once squared, yields

$$\hat{\Sigma}^2 = \frac{1}{2}\left[\left(-\partial_x^2 - \partial_y^2 + n\right)\hat{1} + \frac{i}{2\sqrt{n}}\left(n_x\hat{\sigma}_1\hat{\sigma}_2 + n_y\hat{\sigma}_1\hat{\sigma}_3\right)\right], n_{x,y} = \partial_{x,y}n, \tag{82}$$

and can eventually be written as

$$\hat{T} = \hat{\Sigma}^2 - \frac{1}{4\sqrt{n}}\begin{pmatrix} n_x & n_y \\ n_y & -n_x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \hat{A}^-\hat{A}^+ - (\partial_y^2 + \frac{n_x}{2\sqrt{n}}) & i[\hat{A}^-, \partial_y] - \frac{n_y}{2\sqrt{n}} \\ -i[\hat{A}^+, \partial_y] - \frac{n_y}{2\sqrt{n}} & \hat{A}^+\hat{A}^- - (\partial_y^2 - \frac{n_x}{2\sqrt{n}}) \end{pmatrix}, \tag{83}$$

which yields the possibility of extending supersymmetric treatment to physical problems like the electromagnetic wave propagation in non-homogeneous optical fibers.

We have mentioned the necessity of employing higher-order matrices for the DFM of PDE involving derivatives larger than 2. This is not necessarily true, for example the differential equation regarding the beam vibration [54] in its simpler form writes

$$\partial_t^2 u(x,t) = -\gamma\partial_x^4 u(x,t). \tag{84}$$

In this case, the use of the Pauli matrices eventually yields

$$\hat{\sigma}_1 \partial_t \underline{u} + \hat{\sigma}_2 \sqrt{\gamma} \partial_x^2 \underline{u} = 0 \to \begin{pmatrix} 0 & \partial_t + i\partial_x^2 \\ \partial_t - i\partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0, \tag{85}$$

which indicates that Equation (84) can be factored in a Schröedinger and gravity wave like equations [55].

The problems we treated in this article regard PDEs viewed as evolution equations. In a forthcoming investigation, we will discuss how the boundary conditions can be included in the factorizing procedure we have proposed.

7. Discussion and Conclusions

This article covered different issues regarding the Dirac factorization methods. We pursued a research path opened in refs. [6,13–15], and we found a wealth of new results, which are summarized below. The articles in refs. [13–15] have presented a formulation of DFM applied to both quantum and classical physics and have shown the possibility of merging Pauli matrices and commutation bracket algebras to recover the intrinsic supersymmetric nature of the quantum oscillator and second-order differential equations associated with Schrödinger equation associated with a generic potential.

The article in ref. [6] has proposed a particularly interesting point of view. It extends the DFM to different equations in mathematical physics, showing that the factorized equations (including also those of fractional nature) can be associated with an actual physical interpretation.

The use of the perspectives opened in the quoted references has motivated the application of DFM to the Laplace equation. Therefore, allowing the possibility of framing within a unified concept, the Cauchy–Riemann conditions and the relevant extension to multivariable complex analysis.

This article considered many topics, and some of them were only touched on. We believe the points raised regarding the contiguity between Dirac factorization and fractional calculus deserve deeper understanding. In ref. [56], it has been pointed out that these studies go well beyond the pure mathematical aspects. Indeed, they open the possibility of new directions in field theories [57] and speculations regarding the nature of the intrinsic spin, which within this context does not appear to be a manifestation of relativity [43].

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Appendix A. The Factorization Method and the Relativistic Heat Equation

As previously underscored, the use of the evolution operator method allows us to write the solution of Equation (17) as

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = e^{-it\begin{pmatrix} 1 & -i\partial_\xi \\ -i\partial_\xi & -1 \end{pmatrix}} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}_{t=0} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\tau\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}} \begin{pmatrix} \tilde{\psi}_+(k) \\ \tilde{\psi}_-(k) \end{pmatrix} e^{ik\xi} dk, \tag{A1}$$

where $\tilde{\psi}_{\pm}(k)$ denotes the Fourier transform of $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}_{t=0}$ (recall that the up and down components are functions of the position coordinate only). The exponential matrix in the anti-transform integral of Equation (A1) explicitly writes [25]

$$e^{-i\tau\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}} = \begin{pmatrix} \frac{-2i}{\sqrt{\Delta}}\sin\left(\frac{\sqrt{\Delta}}{2}\tau\right) + \cos\left(\frac{\sqrt{\Delta}}{2}\tau\right) & \sin\left(\frac{\sqrt{\Delta}}{2}\tau\right) \\ \sin\left(\frac{\sqrt{\Delta}}{2}\tau\right) & \frac{2i}{\sqrt{\Delta}}\sin\left(\frac{\sqrt{\Delta}}{2}\tau\right) + \cos\left(\frac{\sqrt{\Delta}}{2}\tau\right) \end{pmatrix}, \tag{A2}$$

with $\triangle = 1 + k^2$. It is interesting to note that, considering the Hamiltonian

$$\hat{H} = \hat{\sigma}_1 pc + \hat{\sigma}_3 mc^2, \tag{A3}$$

associated with Equation (17), the following Heisenberg equations of motion can be derived (see also ref. [13])

$$i\hbar \frac{d}{dt}\hat{\chi} = \hat{s}_1 c[\hat{\chi}, \hat{p}] \rightarrow \frac{d}{dt}\hat{\chi} = \hat{v}_{op} = c\hat{\sigma}_1, \tag{A4}$$

with $\hat{v}_{op}$ being the Dirac velocity operator, whose equation of motion writes

$$\hat{a}_{op} = \frac{d}{dt}\hat{v}_{op} = \frac{2\pi c^2}{\lambda_c}\hat{\sigma}_2. \tag{A5}$$

This is a non-zero acceleration operator, characterizing the oscillatory motion known as Zitterbewegung [58] (see below). Although Equation (A3) is unidimensional, the following equation can be derived for the vector $\vec{\sigma} \equiv (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$

$$\frac{d}{dt}\vec{\sigma} = \vec{\Omega} \times \vec{\sigma}, \; \vec{\Omega} \equiv \frac{4\pi c}{\lambda_c}(\frac{\hat{p}}{mc}, 0, 1). \tag{A6}$$

The above rotation is characterized by a frequency $\omega = \frac{2\pi c}{\lambda_c}$, which corresponds to the already quoted Zitterbewegung motion, regarding the particle (ruled by Equation (17)) between up and down states.

We like to underscore that the Hamiltonian (A3) formally holds in the classical case too, the difference being that p is a canonical momentum. Accordingly, Equation (A4) holds in the classical case too. The physical role of the Pauli matrices in a classical context is doubtful, but, formally, the DFM is correct also if applied to a classical relativistic Hamiltonian. Using classical arguments, we do not recover any acceleration (like that reported in Equation (A5)), and no corresponding Zitterbewegung, which is a purely quantum behavior, not includable within a non-quantum framework. However, it must be recalled that in the Foldy–Wouthuysen representation, the mean spin angular momentum is a constant of motion, thus the analogy with the non-relativistic Pauli theory in that representation is more immediate. Indeed, the parallelism between the Pauli theory and the non-quantum theory of electron interaction is relatively straightforward, if relating the position of the classical particle to the mean position of the quantum particle [59]. The classical limit of the Dirac equation is, however, not a secondary problem, and therefore, for further comments, we refer the reader to the specialized literature [60].

Regarding the handling Equation (36), we like to underscore that different factorizations can be proposed. We first note that, considering the operator identity

$$\partial_t^2 - \kappa c^2 \partial_t = c^2 \partial_x^2, \tag{A7}$$

which solved for ∂_t yields

$$\partial_t = \frac{\kappa c^2 \pm \sqrt{\left(\kappa c^2\right)^2 + 4c^2\partial_x^2}}{2}, \tag{A8}$$

after factorizing the square root, à la Dirac, we obtain

$$\partial_t = \frac{\kappa c^2 \pm \left(\kappa c^2 \hat{\sigma}_1 + 2c\partial_x \hat{\sigma}_3\right)}{2}. \tag{A9}$$

Accordingly, Equation (A7) is transformed into a couple of two-component equations. This method will be discussed in a forthcoming investigation, where it will be applied to the solution of Cattaneo [61] and Telegraphers [62] equations.

Appendix B

Our analysis of multipole solutions of Laplace's equation, using the DFM technique, has indicated unsuspected relationships with different aspects of calculus, including families of special polynomials and of complex analysis. The extension of the method to higher-order derivatives has offered a wider scenario, which has opened an interesting and non-trivial generalization of the elements of discussion raised by the study of the ordinary Laplace equation.

This appendix is intended as a complement to our previous discussion in Sections 3 and 4, where we left open some questions, which, as we will see, are not just computational details.

1. Skew multipoles and the three-dimensional Laplace equation
   The solutions of the three-dimensional Laplace equation are not limited to Equation (52) (denoted by $\psi_n^{N,N}(x,y,z)$) which has been constructed as in terms of normal quadrupoles. The inclusion of the skew parts allows us to infer further solutions

$$\psi_n^{S,S}(x,y,z) = \sum_{r=0}^{n} \binom{n}{r} \psi_{n-r}^{S}(x,-iy)\left(i\sqrt{2}z\right)^r \left|\sin\left(\frac{\pi}{2}r\right)\right|, \tag{A10}$$

$$\psi_n^{N,S}(x,y,z) = \sum_{r=0}^{n} \binom{n}{r} \psi_{n-r}^{N}(x,-iy)\left(i\sqrt{2}z\right)^r \left|\sin\left(\frac{\pi}{2}r\right)\right|, \tag{A11}$$

$$\psi_n^{S,N}(x,y,z) = \sum_{r=0}^{n} \binom{n}{r} \psi_{n-r}^{S}(x,-iy)\left(i\sqrt{2}z\right)^r \left|\cos\left(\frac{\pi}{2}r\right)\right|, \tag{A12}$$

   providing linearly independent solutions, as checked by directly plugging them in Equation (48).
2. Third-order Laplace equation

The solution (61) of Equation (60) is expressed in terms of a new family of multipoles denoted by ${}_{(3,k)}\psi_n^N(x,y)$, $k = 0,1,2$, and their use has noticeable mathematical motivations, as reported below.

The pseudo-exponential functions have been defined by Ricci [63] on the eve of the seventies of the last century, they write

$$E(x;r;j) = \sum_{n=0}^{\infty} \frac{x^{rn+j}}{(rn+j)!}, j = 0,1,\ldots,r-1, \tag{A13}$$

satisfy the identities

$$\left(\frac{d}{dx}\right)^k S(x;r,j) = S(x;r;j-k), \tag{A14}$$

and are solutions of the differential equation

$$\left(\frac{d}{dx}\right)^r S(x;r;j) = S(x;r;j), \tag{A15}$$

accordingly, when $r = 3$, the independent linear solutions of (A15) are $S(x;3;0,1,2)$.

In ref. [64], it has been shown that Polynomials of Appèl type [25] can be defined as

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} N_n(x,y;r;j) = S(-xt;r,j)e^{xt}, \tag{A16}$$

where the polynomials N_n read

$$N_n(x,y;r;j) = n! \sum_{r=0}^{\left\lfloor \frac{n-j}{r} \right\rfloor} \frac{(-1)^r x^{n-rs-j} y^{rs+j}}{(n-rs-j)!(rs+j)!}. \tag{A17}$$

Limiting ourselves to $r = 3$ (namely to the "multipoles" introduced in Equations (55)–(57)), we report below the relevant properties

$$\begin{aligned}
&{}_{(3,3)}\psi_n(x,y) = {}_{(3,0)}\psi_n(x,y),\ {}_{(3,4)}\psi_n(x,y) = {}_{(3,1)}\psi_n(x,y),\ {}_{(3,5)}\psi_n(x,y) = {}_{(3,2)}\psi_n(x,y),\\
&\partial_{x\,(3,j)}\psi_n(x,y) = n_{(3,j)}\psi_{n-1}(x,y), j = 0,1,2,\\
&\partial_{y\,(3,0)}\psi_n(x,y) = -n_{(3,2)}\psi_{n-1}(x,y),\\
&\partial^2_{y\,(3,0)}\psi_n(x,y) = -n(n-1)_{(3,1)}\psi_{n-2}(x,y).
\end{aligned} \tag{A18}$$

The previous identities allow us to write the solution of Equation (60) in terms of linear combinations of multipoles ${}_{(3,j)}\psi_n(x,y)$, namely

$$\psi_k(x,y) = \sum_{j=0}^{2} a_j^{(k)}\,{}_{(3,j)}\psi_n(x,y), k = 1,\ldots,3. \tag{A19}$$

Before closing this appendix, we note that the use of the umbral formalism allows the following extension to the three-dimensional solution.

$${}_{(3,0)}\psi_n(x,y,z) = {}_{(3,0)}\psi_n(x,y) = n! \sum_{s=0}^{n} \frac{{}_{(3,0)}\psi_{n-s}(x,-iy)}{s!(n-s)!}\, {}_3\hat{h}(s)\left(\sqrt[3]{2}z\right)^s,\ {}_3\hat{h}(s) = d_{(3\lfloor \frac{s}{3} \rfloor, s)}. \tag{A20}$$

We have limited the previous result to a particular case; the full solution requires a more careful analysis, which cannot be developed in the space of an appendix and will be discussed in a forthcoming investigation.

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